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THE POWER SUBSTITUTION FOR RINGS OF COMPLEX AND REAL FUNCTIONS ON COMPACT METRIC SPACES

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ABSTRACT. The weak power substitution property for rings of matrices over the ring of functions on a compact metric space X is given in terms of cohomological dimension. A compactum with the ring of complex functions $C(X)$ having the following property is constructed: *the units of* $C(X)$ *are not dense in* $C(X)$ *and they are dense among squares.*

1. INTRODUCTION

The power substitution property for rings was introduced by Goodearl [1]. Then that property was applied to the following topological problem [2]:

Let η, ξ_1 and ξ_2 be complex or real vector bundles on a topological space X. Under what kind of conditions does the equality $\eta \oplus \xi_1 \cong \eta \oplus \xi_2$ imply $m\xi_1 \cong m\xi_2$ for some natural number $m \in \mathbf{N}$?

There is the natural bijection between complex (real) bundles on a compactum X and projective modules over the ring of all complex (real) functions $C(X)$ which is defined by prescribing to a given vector bundle ξ the group of its continuous sections $\Gamma(\xi)$. If the endomorphism ring of $\Gamma(\eta)$ satisfies the power substitution property, then the answer to the above question is affirmative.

A ring R has *the power substitution property* if for any equality $ax + by = 1$ there exists a natural number n such that $a1_n + bM_n(R)$ contains a unit; here $M_n(R)$ denotes the ring of all $n \times n$ matrices over R and 1_n denotes the identity matrix. For a commutative ring R this property is the same as the following: whenever $aR + I = R$ for some ideal I there exists a natural n such that $a1_n + M_n(I)$ contains a unit. For non-commutative rings the second property leads to the definition of *the weak power substitution property.* It's easy to check that the power substitution implies the weak power substitution. These definitions are taken from [3].

In this paper we show that the ring $M_n(C(X))$ has the weak power substitution property if and only if the compactum X has cohomological dimension with respect to rationals less than or equal to three in the case of real-valued functions: $\dim_{\mathbf{Q}} X \leq 3$. In the case of complex-valued functions it's equivalent to $\dim_{\mathbf{Q}} X \leq 1$. This result is based on the main results of [3]. For $n = 1, 2$ it

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was proved in [1]. The main result of this paper answers the following question by R. Camps: Does there exist an m-dimensional compactum Y with $m \geq 2$ such that $C_{\mathbb{C}}(Y)$ has the power substitution property with fixed n for all a, b? The answer is positive. In this case we shall call our ring $C_{\mathbb{C}}(Y)$ having the *n-power subsitution property.* We construct a compactum Y of that sort with the property: the units are nowhere dense in $C = C_{\mathbb{C}}(Y)$ and they are dense in C^2, i.e. for every complex function f the square f^2 can be approximated by units, i.e. functions missing zero in $\mathbb{C}$.

Still there is an open question to characterize compact metric spaces whose matrix rings of real (complex) functions have the power substitution (n-power substitution) property. In particular does the weak power substitution property imply a power substitution for that kind of rings?

2. Weak power substitution

Denote by $p_m: SO(n) \to SO(nm)$ the composition of taking the m-th power of matrices and the natural inclusion $SO(n) \to S(nm)$. Similarly, for the complex case denote its composition by $q_m: SU(n) \to SU(nm)$, and denote taking the m-th power for the circle by $(-)^m: S^1 \to S^1$.

The following two propositions define the weak power substitution property for matrix rings over functional spaces in topological terms [3].

Proposition 1. *For a compact space X the matrix ring $M_n(C_{\mathbb{R}}(X))$ over real functions has the weak power substitution property if and only if*

(a) *For every partial map $f: A \to SO(n)$ there is a natural number m and an extension $\overline{f}: X \to SO(nm)$ of the map $p_m \circ f$.*

Proposition 2. *For a compact space X the matrix ring $M_n(C_{\mathbb{C}}(X))$ over complex functions has the weak power substitution property if and only if*

(b) *For every partial map $g: A \to S^1$ there is a number m and an extension of the composition g and $(-)^m$, and*

(c) *For every partial map $f: A \to SU(n)$ there is a number m and an extension $\overline{f}: X \to SU(nm)$ of the composition f and q_m.*

By a *partial map* of a topological space X we mean an arbitrary continuous map of a closed subset of X. Let X and M be topological spaces, and denote by $X\tau M$ the following extension property: *for every partial map $f: A \to M$, there exists an extension $\overline{f}: X \to M$.* The other notation for that is $M \in AE(X)$ which is to be read "M is an absolute extensor for a space X". Let N be a subspace of M. Then it's quite natural to denote by $X\tau(M, N)$ the above extension property where mappings f with the ranges restricted to N are considered. The next generalization of $X\tau(M, N)$ is to consider an arbitrary map $g: N \to M$ instead of an imbedding. Let us denote that extension property by $X\tau g$. For decent spaces M and N the last extension property $X\tau g$ is equivalent to the previous $X\tau(M_g, N)$ where M_g denotes the mapping cylinder of the map g and $N \subset M_g$ is the natural inclusion of the domain N in the mapping cylinder.

By $K(G, n)$ we denote an Eilenberg-MacLane complex for a group G. I would like to recall that the extension property $X\tau K(G, n)$ means precisely that the cohomological dimension of X is less than or equal to n [4], [5]. So, that extension property can serve as a definition of cohomological dimension with respect to a group G. Notation is $\dim_G X$.

Let $SO_{(0)}$, $SO(n)_{(0)}$, $SU_{(0)}$ and $SU(n)_{(0)}$ be localizations at 0 of spaces SO, $SO(n)$, SU and $SU(n)$ [10]. Let h_k denote a map generated by taking the k-th power in SO. Note that $SO_{(0)}$ is the direct limit of the sequence $M_1 \to M_2 \to M_3 \to \cdots$ where the k-th space M_k is the union of mapping cylinders of $h_i\colon SO \to SO$ with $i \le k$, which is called a telescope of $(h_i : i \le k)$, and the k-th bonding map is the inclusion of M_k in M_{k+1}. The inclusion of $SO(n)$ in SO defines the natural inclusion of $SO(n)_{(0)} \subset SO_{(0)}$ where $SO(n)_{(0)}$ is the direct limit of a corresponding subsequence of telescopes N_i. There is a similar construction for the pair $(SU_{(0)}, SU(n)_{(0)})$. Note that the same construction works for the circle and the space $S^1_{(0)}$ is the Eilenberg-MacLane space $K(\mathbb{Q}, 1)$.

Lemma 3. *The conditions* (a)–(c) *of Propositions* 1, 2 *are equivalent respectively to*:

(a′) $X\tau(SO_{(0)}, SOn_{(0)})$,
(b′) $\dim_{\mathbb{Q}} X \le 1$,
(c′) $X\tau(SU_{(0)}, SU(n)_{(0)})$.

Proof. (a) $\Rightarrow$ (a′). Let $f\colon A \to SO(n)_{(0)}$ be a partial map. Since A is compact, there is a number i such that $f(A) \subset N_i$. Property (a) implies that there is a number m such that the composition $p_m \circ \pi \circ f$ has an extension where $\pi\colon N_i \to SO(n)$ is the projection of the telescope onto the image of the last mapping. Note that $r \circ p_m = i_n \circ h_{i+m} \circ \cdots \circ h_{i+1}$ for some $r\colon SO(nm) \to SO$ and the natural inclusion $i_n\colon SO(n) \to SO$. The map f is homotopic to $i_n \circ h_{i+m} \circ \cdots \circ h_{i+1} \circ \pi \circ f$, hence by the Homotopy Extension Theorem f has an extension over X to the map to M_{i+m}. Therefore f has an extension to $SO_{(0)}$.

(a′) $\Rightarrow$ (a). Let $f\colon A \to SO(n)$ be a partial map. We may assume that $SO(n)$ is the first space in our telescope construction for $SO(n)_{(0)}$. Let $\overline{f}\colon X \to SO_{(0)}$ be an extension of f. Since X is compact, there is a number k such that $\overline{f}(X) \subset M_k$ and even more, it lies in some k-telescope L_k of $SO(l)$'s contained in M_k. If $l \le k$, then (a) is checked with $m = k$. If $l \ge k$, we consider the projection on the l-level in our direct system and then (a) holds with $m = l$.

A compactness of X easily implies the equivalence (b) $\Leftrightarrow$ (b′).

The proof of (c) $\Leftrightarrow$ (c′) is similar. □

Lemma 4. *For finite dimensional compacta the property* (b′) *implies* (c′); *and* (a′) *is equivalent to* (a″): $\dim_{\mathbb{Q}} X \le 3$, *provided* $n \ge 3$.

Proof. Show that (b′) implies $X\tau SU_{(0)}$. According to Theorem 10 of [7] $X\tau SU_{(0)}$ is equivalent to $\dim_{\pi_i} X \le i$ where $\pi_i = \pi_i(SU_{(0)})$ is a vector space over $\mathbb{Q}$. So, since only the third (and higher) homotopy group of $SU_{(0)}$ is not trivial, $X\tau SU_{(0)}$ is equivalent to $\dim_{\mathbb{Q}} X \le 3$ and (b′) implies that.

According to Bott's theorem $\pi_i(SO_{(0)}) = \mathbb{Q}$ if $i = 8k+3$ or $i = 8k-1$ and $= 0$ otherwise. Then again by Theorem 10 of [7] (a″) implies (a′). In order to complete the proof we need a relative version of Theorem 6 of [7]. □

We recall that SP^nY denotes the n-th symmetric power of the space Y. For a pointed space Y there is a natural imbedding $SP^nY \to SP^{n+1}Y$ for any

n. We define the infinite symmetric power of Y as the direct limit $SP^\infty = \lim SP^n Y$.

Theorem 5. *For any compact metric space X, every ANR-pair (M, N), and for every $i = 1, 2, \dots, \infty$, the property $X\tau(M, N)$ implies $X\tau(SP^i M, SP^i N)$.*

Proof. The proof of the theorem does not differ much from that of Theorem 6 of [7]. So we omit it and note that with a slight modification the proof works for non-compact metric spaces.

Completion of the proof of Lemma 4. Apply Theorem 5 to obtain

$$X\tau(SP^\infty(SO_{(0)}),\ SP^\infty(SO(n)_{(0)})).$$

According to a theorem of Dold-Thom [8], [9] for every ANR-space M we have a homotopy equivalence $SP^\infty M \sim \lim_m \prod_{i=1}^m K(H_i(M), i)$, where $H_i(M)$ means the i-dimensional integer homology group of M. The integer homology groups of $SO_{(0)}$ coincide with the rational homology groups of SO which are equal to rational homologies of the product $S^3 \times S^5 \times S^7 \times \cdots$. For $n \geq 3$ the same is true only with the finite product for homologies of $SO(n)_{(0)}$. Note that in dimension 3 the inclusion $SO(n)_{(0)} \to SO_{(0)}$ induces an isomorphism in homologies $(n \geq 3)$.

So, there are homotopy equivalences $h_1\colon K(\mathbb{Q}, 3) \times N_1 \to SP^\infty(SO(n))_{(0)}$ and $h_2\colon SP^\infty(SO_{(0)}) \to K(\mathbb{Q}, 3) \times M_1$, where M_1 and N_1 are 3-connected spaces, and the composition $h = h_2 \circ i_n \circ h_1$ induces an isomorphism of 3-dimensional homotopy groups. Hence the restriction of the composition $w \circ h$ of h and the projection onto the first factor $w\colon K(\mathbb{Q}, 3) \times M_1 \to K(\mathbb{Q}, 3)$ induces an isomorphism of 3-dimensional homotopy groups and therefore $h' = w \circ h$ is a homotopy equivalence.

Now let $f\colon A \to K(\mathbb{Q}, 3) = K(\mathbb{Q}, 3) \times \{pt\}$ be a partial map. By the assumption and Theorem 5 the map $i_n \circ h_1 \circ f$ has an extension $\overline{f}\colon X \to SP^\infty(SO_{(0)})$. Note that $w \circ \overline{f}$ restricted on A coincides with $h' \circ f$. Denote by g' a homotopy inverse map to h'. We know that the map $h' \circ f$ is extendable. Then the map $g' \circ h' \circ f$ is extendable. Since $g' \circ h'$ is homotopic to the identity map, the composition $g' \circ h' \circ f$ is homotopic to f. The Homotopy Extension Theorem implies that there exists an extension of f. Since f is an arbitrary partial map, we have $\dim_{\mathbb{Q}} X \leq 3$. □

We summarize the above in the following theorems.

Theorem 6. *Let X be a compact metric space. Then the following are equivalent*:

(1) $M_n(C_{\mathbb{R}}(X))$ *has the weak power substitution for some* $n \geq 3$.
(2) $\dim_{\mathbb{Q}} X \leq 3$.
(3) $M_n(C_{\mathbb{R}}(X))$ *has the weak power substitution for all* n.

Note that $M_n(C_{\mathbb{R}}(X))$ always has the power substitution for $n = 1, 2$ [2], [3].

Theorem 7. *Let X be a compact metric space. Then the following are equivalent*:

(1) $M_n(C_{\mathbb{C}}(X))$ *has the weak power substitution for some* n.
(2) $\dim_{\mathbb{Q}} X \leq 1$.
(3) $M_n(C_{\mathbb{C}}(X))$ *has the weak power substitution for all* n.

3. n-POWER SUBSTITUTION

Let $(-)^n\colon S^1 \to S^1$ be a map of the circle to itself of degree n (taking the n-th power). It follows by [3] that the ring of complex functions on a compactum X has n-power substitution if and only if $X\tau(-)^n$.

Theorem 8. *For every n and every prime p there exist an n-dimensional compactum X with the p^{n-1}-power substitution of $C_{\mathbb{C}}(X)$.*

Corollary 9. *There exist a compactum X such that the space of units $UC_{\mathbb{C}}(X)$ is not dense in $C_{\mathbb{C}}(X)$ and $UC_{\mathbb{C}}(X)$ is dense in $C_{\mathbb{C}}(X)^2 = \{f^2 : f \in C_{\mathbb{C}}(X)\}$.*

Proof. The compactum X of the theorem for $n = p = 2$ has the requisite properties. First, we recall that if $UC_{\mathbb{C}}(X)$ is dense in $C_{\mathbb{C}}(X)$, then $\dim X \leq 1$ [6]. Since $\dim X \geq 2$, the units are not dense in $C_{\mathbb{C}}(X)$. Note that X has the property $X\tau(-)^2$.

Let $f\colon X \to \mathbb{C}$ and $\varepsilon > 0$ be given; we construct $g\colon X \to \mathbb{C} - \{0\}$ to be ε-close to f^2. Let $S^1_\varepsilon = \{z \in \mathbb{C}\colon |z| = \varepsilon\}$ be the circle bounding the disk D_ε in the complex plane $\mathbb{C}$. Denote $A = f^{-1}(S^1_\varepsilon)$ and $Y = f^{-1}(D_\varepsilon)$. According to the property $X\tau(-)^2$ there is an extension of $(-)^2 \circ f|_A\colon A \to S^1_{\varepsilon^2}$ over Y to a map $p\colon Y \to S^1_{\varepsilon^2}$. Glue maps $f^2|_{X-Y}$ and p to obtain a map $p'\colon X \to \mathbb{C} - \{0\}$ which is ε-close to f^2. □

Let (X, Y) be a pair of spaces, $Y \subset X$, and let $g\colon L \to M$ be a map. Denote by $(X, Y)\tau g$ the property that for every map $f\colon Y \to L$ there is a map $q\colon X \to M$ such that q is an extension of the composition $g \circ f$.

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ L & \xrightarrow{g} & M \end{array}$$

For every $h\colon Z \to X$ we denote the pair $(h^{-1}(X), h^{-1}(Y))$ by $h^{-1}(X, Y)$.

We need the following lemmata.

Lemma 10. *For any simplicial complex (L, π) over the n-dimensional simplex σ^n and for every prime p there is a map of an n-dimensional polyhedron $\xi\colon R \to L$ such that:*

(1) *for every simplex $\Delta \subset L$ the property $\xi^{-1}(\Delta, \partial\Delta)\tau(-)^p$ holds, where $(-)^p\colon S^1 \to S^1$ denotes taking the p-th power on the unit circle,*

(2) *$\xi\colon H^n(L, \mathbb{Z}_p) \to H^n(R; \mathbb{Z}_p)$ is an isomorphism.*

By the definition in [11] a simplicial complex over the simplex σ is a polyhedron L with a simplicial map $\pi\colon L \to \sigma$ such that the restriction of π on every simplex of L is injective. Note that the first barycentric subdivision of any triangulation of the n-dimensional polyhedron L defines a structure of simplicial complex over the n-simplex.

Proof. A similar lemma is proved in [5].

We apply induction on n. For $n = 1$ we may define $R = L$ and $\xi = \mathrm{id}$.

In order to make the inductive step from n to $(n+1)$ we need to construct that resolution for the $(n+1)$-dimensional simplex σ. Then for an arbitrary simplicial complex (L, π) over the $(n+1)$-simplex we may apply a pull-back construction.

Assume that $\xi_Y\colon Y \to \partial\sigma$ is such a resolution for the boundary of the $(n+1)$-simplex with some triangulation turning it into a complex over the n-simplex. Let $\alpha_1, \dots, \alpha_m$ be a basis of the 1-dimensional cohomology group $H^1(Y)$. By ϕ_i we denote a map $\phi_i\colon Y \to S^1$ which generates the element $p\alpha_i$. Let $T^m = S^1 \times \dots \times S^1$ be an m-dimensional torus and let T^m_n denote the n-skeleton of T^m with respect to the natural cellular structure. The diagonal product $\Delta\phi_i\colon Y \to T^m$ has a cellular approximation $\phi\colon Y \to T^m_n$. Let M_ϕ be a mapping cylinder, and define a map $\xi\colon M_\phi \to \sigma$ as the natural linear extension of ξ_Y.

First note that the property (1) holds. To check the property (2) we consider the diagram:

$$\begin{array}{ccccccc}
H^n(R^{[n]};\mathbb{Z}_p) & \longrightarrow & H^{n+1}(R/R^{[n]};\mathbb{Z}_p) & \longrightarrow & H^{n+1}(R;\mathbb{Z}_p) & \longrightarrow & 0 \\
\alpha\big\uparrow & & \beta\big\uparrow & & \xi^*\big\uparrow & & \\
H^n(L^{(n)};\mathbb{Z}_p) & \longrightarrow & H^{n+1}(L/L^{(n)};\mathbb{Z}_p) & \longrightarrow & H^{n+1}(L;\mathbb{Z}_p) & \longrightarrow & 0
\end{array}$$

Here $L^{(n)}$ denotes the n-dimensional skeleton of a simplicial complex L over the $(n+1)$-simplex σ and $R^{[n]}$ denotes its preimage under ξ. Note that α is an isomorphism by the induction assumption. The quotient space $R/R^{[n]}$ is homeomorphic to the wedge of cones $\bigvee_\sigma C_\phi$ the map ϕ. Hence, to complete the proof by applying the five-lemma it is sufficient to show that the projection $\gamma\colon C_\phi \to S^{n+1}$ induces an isomorphism of cohomology groups in dimension $n+1$. Since the map $\Delta\phi_i$ induces a zero homomorphism $\bmod p$ in dimension 1, due to the ring structure on $H^*(T^m)$ it induces a zero homomorphism for n-dimensional cohomology $\bmod p$. Whence $\phi^*\colon H^n(T^m_n;\mathbb{Z}_p) \to H^n(Y;\mathbb{Z}_p)$ is a zero homomorphism. The Puppe exact sequence gives a diagram:

$$\begin{array}{ccccccc}
H^{n+1}(\Sigma T^m_n;\mathbb{Z}_p) & \xrightarrow{\Sigma\phi^*=0} & H^{n+1}(\Sigma Y;\mathbb{Z}_p) & \longrightarrow & H^{n+1}(C_\phi;\mathbb{Z}_p) & \longrightarrow & 0 \\
\big\uparrow & & \Sigma\xi^*\big\uparrow & & \gamma^*\big\uparrow & & \\
0 = H^{n+1}(\Sigma pt;\mathbb{Z}_p) & \longrightarrow & H^{n+1}(\Sigma S^n;\mathbb{Z}_p) & \longrightarrow & H^{n+1}(S^{n+1};\mathbb{Z}_p) & \longrightarrow & 0
\end{array}$$

Since ξ_Y^* is an isomorphism in the dimension n, γ^* is also an isomorphism. □

Lemma 11. *Let a compactum $X = \lim\{P_i; q_i^{i+1}\}$ be a limit space of the inverse system of n-dimensional polyhedra provided with triangulations τ_i such that*

(1) $\lim mesh q_k^{k+i}(\tau_{k+i}) = 0\ \forall k$,

(2) $(q_i^{i+1})^{-1}(L)$ *is a subpolyhedron of P_{i+1} with respect to τ_{i+1} for every subcomplex $L \subset P_i$,*

(3) $((q_i^{i+1})^{-1}(\sigma, \partial\sigma))\tau(-)^p$ *holds for every simplex $\sigma \in \tau_i$.*

Then $X\tau(-)^{p^{n-1}}$.

Proof. Let A be a closed subset of X and $\phi\colon A \to S^1$ be an arbitrary map. Since S^1 is an ANR, for large enough i there is a subcomplex $A_i \subset P_i$ with respect to the triangulation τ_i and a map $\phi_i\colon A_i \to S^1$ such that the restriction $\phi_i \circ q_i^\infty|_A$ is homotopic to ϕ. Since S^1 is path-connected, there is an extension $\phi'\colon P^{(1)} \cup A_i \to S^1$ of ϕ_i. Let $\psi_1 = \phi' \circ q_i^{i+1}$ and $W_n = (q_i^{i+1})^{-1}(P_i^{(n)} \cup A_i)$. Apply the condition (3) to all 2-dimensional simplexes σ to obtain a map

$\psi_2\colon W_2 \to S^1$ such that $(-)^p \circ \psi_1 = \psi_2|_{W_1}$. Then apply the condition (3) to all 3-simplexes to obtain a map $\psi_3\colon W_3 \to S^1$ such that $(-)^p \circ \psi_2 = \psi_3|_{W_2}$ and so on. Then the map $(-)^{p^{n-1}}\psi_1 = \psi_n$. This means that the composition $(-)^{p^{n-1}} \circ \phi_i$ has an extension over P_{i+1}. Hence $(-)^{p^{n-1}} \circ \phi_i \circ q_i^\infty|_A$ has an extension and by the Homotopy Extension Theorem the map $(-)^{p^{n-1}} \circ \phi$ has an extension. □

Proof of Theorem 8. By induction we construct an inverse system $\{P_i; q_i^{i+1}\}$ with the properties (1)–(3) of Lemma 11 such that q_i^{i+1} induces an isomorphism of the n-dimensional cohomology group with $\mathbb{Z}_p$ coefficients for every i. We take $P_1 = S^n$ and consider an arbitrary triangulation τ_1 with $mesh < 1$ which turns P_1 into a simplicial complex over the n-simplex. Then we apply Lemma 10 to construct $q_1^2\colon P_2 \to P_1$ and define a triangulation τ_2 on P_2 such that $mesh(\tau_2)$, $mesh(q_1^2(\tau_2)) < 1/2$ and so on. According to Lemma 11 X has the property $X\tau(-)^{p^{n-1}}$. The dimension of $X \le n$ since all P_i are n-dimensional. Since q_1^∞ induces an isomorphism of the n-dimensional mod p cohomology groups, X has n-dimensional mod p cohomology non-trivial and hence is at least n-dimensional. Thus $\dim X = n$. □

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