GLOBAL ATTRACTIVITY FOR A POPULATION MODEL WITH TIME DELAY

JOSEPH W.-H. SO AND J. S. YU

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Abstract. In this paper we give a sufficient condition which guarantees the global attractivity of the zero solution of a population growth equation.

1. Introduction

The delay differential equation
\[ N'(t) = \beta N(t) \left[ 1 - \frac{N(t-q)}{K} \right], \quad q > 0, \]
was proposed by Hutchinson [1] as a model for the growth of a single species. Using the change of variables \( x(t) = \frac{N(qt)}{K} - 1 \) and defining \( r = \beta q \), the above "Hutchinson's equation" can be reduced to
\[ x'(t) = -rx(t-1)[1+x(t)] \]
(cf. Wright [4]).

More realistically, we may assume that the growth rate \( r \) depends on time. Thus consider the delay differential equation
\[ y'(t) = -r(t)[1+y(t)]y(t-1), \quad t \geq 0, \]
where \( r : [0, \infty) \to (0, \infty) \) is continuous. This equation has been studied by Sugie [3] where it was shown that the zero solution of (1.1) is uniformly stable provided there exists a constant \( 0 < r_0 < \frac{3}{2} \) such that \( r(t) \leq r_0 \) for all \( t \geq 0 \).

In this paper, we are interested in the global attractivity of the zero solution. Due to the biological interpretation of (1.1), we are only interested in solutions \( y(t) \) of (1.1) such that \( y(t) \geq -1 \) for \( -1 \leq t \leq 0 \) and \( y(0) > -1 \). In that case,
\[ 1 + y(t) = (1 + y(0))e^{-\int_0^t r(s)y(s-1)ds} > 0 \]
and so \( y(t) > -1 \) for all \( t \geq 0 \).

We now state our main result.

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Theorem 1.1. If
\[ \int_{t-1}^{t} r(s) \, ds \leq \frac{3}{2} \quad \text{for all large } t \]
and
\[ \int_{0}^{\infty} r(s) \, ds = \infty, \]
then every solution of (1.1) tends to zero as \( t \to \infty. \)

In the case when \( r(t) \) is a constant \( r_0 \) and \( 0 < r_0 < \frac{3}{2} \), Theorem 1.1 was proved in Wright [4] (see also Kuang [2, Theorem 2.1, p.120]). It was conjectured in [4] (see also [2, Open Problem 4.1, p.171]) that the result is still true for \( 0 < r_0 < \frac{3}{2} \). This question remains open. In [2], it was conjectured (Open Problem 4.2, p.171) that under the assumption \( 0 < r(t) < \frac{3}{2} \), every solution of (1.1) tends to zero. Theorem 1.1 gives a positive answer to this conjecture, provided (1.3) holds.

2. Preliminaries

Lemma 2.1. Let \( 0 < \alpha < \frac{1}{2} \). The system of inequalities
\[ \begin{align*}
    u &\leq e^{\alpha - u^2} - 1, \\
    v &\leq 1 - e^{-u - \alpha u^2}
\end{align*} \]
has a unique solution: \( (v, u) = (0, 0) \) in the nonnegative quadrant \( \{(v, u) : v > 0, u > 0\} \).

Proof. Assume that (2.1) has another solution in the first quadrant of the \( v-u \) plane besides (0,0), say \( (v_0, u_0) \). Then \( u_0 > 0 \) and \( 0 < v_0 < 1 \). Define \( V_1 \) to be the curve: \( u = e^{v - \alpha v^2} - 1 \) and \( V_2 \) to be the curve: \( v = 1 - e^{-u - \alpha u^2} \).

Clearly \( \frac{dv}{du}|_{(0,0)} = 1, \frac{du}{dv}|_{(0,0)} = 1 - 2\alpha \) and \( \frac{dv}{du}|_{(0,0)} = 1 - 6\alpha \) for \( V_1 \). On the other hand, \( \frac{dv}{du}|_{(0,0)} = 1, \frac{du}{dv}|_{(0,0)} = 1 - 2\alpha \) and \( \frac{dv}{du}|_{(0,0)} = 12\alpha^2 - 6\alpha + 2 \) for \( V_2 \).

Hence \( V_2 \) lies above \( V_1 \) near \( (0,0) \). The existence of \( (v_0, u_0) \) implies that the curves \( V_1 \) and \( V_2 \) must intersect at a point in the first quadrant besides \( (0,0) \). Let \( (v_1, u_1) \) be the first such point, i.e. \( v_1 \) is smallest. Then the slope of \( V_1 \) at \( (u_1, v_1) \) is no less than the slope of \( V_2 \) at \( (u_1, v_1) \), i.e.
\[ (1 - 2\alpha v_1)e^{v_1 - \alpha v_1^2} \geq \frac{1}{1 + 2\alpha u_1}e^{u_1 + \alpha u_1^2} \]
or
\[ (1 - 2\alpha v_1)(1 + 2\alpha u_1) \geq e^{u_1 - v_1 + \alpha(u_1^2 + v_1^2)}. \]

Claim: \( u_1 > v_1 \).

Proof. Let \( \phi(x) = 1 - e^{-x - \alpha x^2} - x \). Then \( \phi(0) = 0 \) and \( \phi'(x) < 0 \) for \( x > 0 \), since \( 2\alpha < 1 \). Thus \( \phi(x) < 0 \) for \( x > 0 \) and \( v_1 = \phi(u_1) + u_1 < u_1 \).

Using the inequality \( e^x > 1 + x \ (x > 0) \), we have
\[ 1 + 2\alpha(u_1 - v_1) - 4\alpha^2 u_1 v_1 > 1 + u_1 - v_1 + \alpha(u_1^2 + v_1^2) \]
or
\[ (-1 + 2\alpha)(u_1 - v_1) - 4\alpha^2 u_1 v_1 > \alpha(u_1^2 + v_1^2) \]
which is a contradiction, since \( 0 < \alpha < \frac{1}{2} \). This completes the proof.
Lemma 2.2. If (1.3) holds, then every nonoscillatory solution of (1.1) tends to zero as \( t \to \infty \).

Proof. Let \( y(t) \) be an nonoscillatory solution of (1.1). Then there exists \( t_0 \) such that \( y(t) \) is of one sign for \( t \geq t_0 \). Consider first the case \( y(t) \leq 0 \) for \( t \geq t_0 \). Since \( 1 + y(t) \geq 0 \), by (1.1) \( y'(t) \geq 0 \). Thus \( y(t) \) is increasing and \( \lim_{t \to \infty} y(t) = -c \) exists. Clearly \( -1 < y(t_0) \leq -c \leq 0 \). Integrating (1.1), we get (for \( t > t_0 + 1 \))

\[
-\ln(1 + y(t)) + \ln(1 + y(t_0 + 1)) = \int_{t_0+1}^{t} r(s)y(s-1)\,ds \leq -c \int_{t_0+1}^{t} r(s)\,ds.
\]

By (1.3), the right-hand side tends to \(-\infty\) as \( t \to \infty \) unless \( c = 0 \). On the other hand, the left-hand side has a finite limit; therefore \( c = 0 \). Hence \( \lim_{t \to \infty} y(t) = 0 \). The case when \( y(t) \) is eventually nonnegative is similar. This completes the proof.

The following lemma was essentially proved in [2, Theorem 3.1, p.128].

Lemma 2.3. Assume (1.2) holds. Let \( y(t) \) be an oscillatory solution of (1.1). Then \( y(t) \) is bounded above and is bounded below from \(-1\) for \( t \geq 0 \).

Proof. Let \( t_0 > 0 \) be large enough so that (1.2) holds for all \( t \geq t_0 \). Let \( t^* \) be a local maximum point of \( y(t) \) \( (t \geq t_0 + 1) \). Then \( y'(t^*) = 0 \) and by (1.1) \( y(t^* - 1) = 0 \). Integrating (1.1) from \( t^* - 1 \) to \( t^* \), we have

\[
1 + y(t^*) = e^{-\int_{t^*-1}^{t^*} r(s)y(s-1)\,ds}.
\]

Since \( y(s-1) \geq -1 \), by (1.2)

\[
1 + y(t^*) \leq e^{\int_{t^*-1}^{t^*} r(s)\,ds} \leq e^{\frac{1}{2}}
\]

and \( y(t^*) \leq e^{\frac{1}{2}} - 1 \). Consequently, \( \limsup_{t \to \infty} y(t) \leq e^{\frac{1}{2}} - 1 \).

Next, let \( t_* \) be a local minimum point of \( y(t) \) \( (t \geq t_0 + 3) \). Then \( y'(t_*) = 0 \) and \( y(t_* - 1) = 0 \). Proceeding similarly as before and using the fact that \( y(s-1) \leq e^{\frac{1}{2}} - 1 \), we have

\[
1 + y(t_*) \geq e^{\int_{t_*-1}^{t_*} r(s)(1-e^{\frac{1}{2}})\,ds} = e^{-(e^{\frac{1}{2}} - 1) \int_{t_*-1}^{t_*} r(s)\,ds} \geq e^{-(e^{\frac{1}{2}} - 1)\frac{1}{2}}.
\]

Hence,

\[
y(t_*) \geq e^{-\frac{2(e^{\frac{1}{2}} - 1)}{3}} - 1,
\]

and \( \liminf_{t \to \infty} y(t) \geq e^{-\frac{1}{2}(e^{\frac{1}{2}} - 1)} - 1 > -1 \).

3. PROOF OF THEOREM 1.1

To complete the proof of Theorem 1.1, all we need is to show

Lemma 3.1. If (1.3) holds, then every oscillatory solution of (1.1) tends to zero as \( t \to \infty \).
Proof. Let $y(t)$ be an oscillatory solution of (1.1). By Lemma 2.3 $y(t)$ ($t \geq 0$) is bounded above and bounded below away from $-1$. Let

$$u = \limsup_{t \to \infty} y(t), \quad -v = \liminf_{t \to \infty} y(t).$$

Then $0 \leq v < 1$ and $0 \leq u < \infty$. It suffices to show that $u = v = 0$. For any $\epsilon$, choose $t_0 = t_0(\epsilon)$ such that

$$v_1 \equiv -v - \epsilon < y(t - 1) < u + \epsilon \equiv u_1, \quad \text{for } t \geq t_0.$$

We assume that $\epsilon$ is small enough so that $0 < v_1 < 1$ and that $t_0$ is large enough so that (1.2) holds for $t \geq t_0 - 2$. Using (1.1), we have

$$y'(t) \leq r(t)(1 + y(t))v_1, \quad t \geq t_0,$$

and

$$y'(t) \geq -r(t)(1 + y(t))u_1, \quad t \geq t_0.$$

Let \( \{t^*_n\} \) be an increasing sequence such that \( t^*_n \geq t_0 + 1, \ y'(t^*_n) = 0, \lim_{n \to \infty} t^*_n = \infty \) and \( \lim_{n \to \infty} y(t^*_n) = u \). By (1.1), \( y(t^*_n - 1) = 0 \). For \( t \in [t^*_n - 1, t^*_n] \), we can integrate (3.3) from \( t - 1 \) to \( t^*_n - 1 \) and get

$$-\ln[1 + y(t - 1)] \leq v_1 \int_{t - 1}^{t^*_n - 1} r(s) \, ds$$

or

$$y(t - 1) \geq -1 + e^{-v_1 \int_{t - 1}^{t^*_n - 1} r(s) \, ds} \quad \text{for } t \in [t^*_n - 1, t^*_n].$$

By (1.1) it follows that

$$y'(t) \leq r(t)[1 + y(t)] \left[ 1 - e^{-v_1 \int_{t - 1}^{t^*_n - 1} r(s) \, ds} \right], \quad t \in [t^*_n - 1, t^*_n].$$

Combining this with (3.3), we have

$$\left( \ln[1 + y(t)] \right)' \leq \min \left\{ r(t)v_1, r(t) \left[ 1 - e^{-v_1 \int_{t - 1}^{t^*_n - 1} r(s) \, ds} \right] \right\}, \quad t \in [t^*_n - 1, t^*_n].$$

We will prove that

$$\ln[1 + y(t^*_n)] \leq v_1 - \frac{1}{6} v_1^2.$$

There are two possibilities.
Case 1: \( \int_{t_1}^{t_2} r(s) \, ds \leq -\frac{\ln(1-v_1)}{v_1} \). Then by (3.5)

\[
\ln[1 + y(t^n)] \\
\leq \int_{t_1}^{t_2} r(t) \left[ 1 - e^{-v_1 \int_{t_1}^{t_2} r(s) \, ds} \right] dt \\
= \int_{t_1}^{t_2} r(t) \left[ 1 - e^{-v_1 \int_{t_1}^{t_2} r(s) \, ds} - v_1 \int_{t_1}^{t_2} r(s) \, ds \right] dt \\
\leq \int_{t_1}^{t_2} r(t) \left[ 1 - e^{-\frac{1}{2}v_1 e^v_1 \int_{t_1}^{t_2} r(s) \, ds} \right] dt \\
= \int_{t_1}^{t_2} r(t) dt - e^{-\frac{1}{2}v_1} \int_{t_1}^{t_2} r(t) e^{-v_1 \int_{t_1}^{t_2} r(s) \, ds} dt \\
= \int_{t_1}^{t_2} r(t) dt - e^{-\frac{1}{2}v_1} \left[ e^{-v_1 \int_{t_1}^{t_2} r(s) \, ds} - 1 \right] \\
= \int_{t_1}^{t_2} r(t) dt - e^{-v_1} \left[ 1 - e^{-v_1 \int_{t_1}^{t_2} r(s) \, ds} \right] \\
= \int_{t_1}^{t_2} r(t) dt - \frac{1}{v_1} \ln(1 - v_1) - e^{-v_1} \left[ 1 - e^{-v_1 \int_{t_1}^{t_2} r(s) \, ds} \right].
\]

The function \( \phi(x) = x - \frac{1}{v_1} e^{-v_1 (\frac{1}{2} - x)} (1 - e^{-v_1 x}) \) is increasing for \( 0 \leq x \leq \frac{3}{2} \).

Thus for \( \int_{t_1}^{t_2} r(t) \, dt \leq -\frac{\ln(1-v_1)}{v_1} \leq -\frac{3}{2} \), we have

\[
\ln[1 + y(t^n)] \leq \ln(1 - v_1) - \frac{1}{v_1} e^{-v_1 (1 + \ln(1 - v_1))} \ln(1 - v_1) \\
= -\frac{\ln(1 - v_1)}{v_1} - e^{-v_1 (1 + \ln(1 - v_1))}.
\]

Using the fact \( e^{-x} > 1 - x \) for \( x > 0 \), we have

\[
\ln[1 + y(t^n)] \leq -1 + \frac{3}{2} v_1 - \frac{(1 - v_1) \ln(1 - v_1)}{v_1} < v_1 - \frac{1}{6} v_1^2,
\]

according to (2.21) on p.123 of [2].

For \( \int_{t_1}^{t_2} r(t) \, dt \leq \frac{3}{2} \leq -\frac{\ln(1-v_1)}{v_1} \), we have

\[
\ln[1 + y(t^n)] \leq \int_{t_1}^{t_2} r(t) \, dt - \frac{1}{v_1} \left[ e^{-\frac{1}{2}v_1} e^{v_1 \int_{t_1}^{t_2} r(t) \, dt} - e^{-\frac{1}{2}v_1} \right].
\]

The function \( x \mapsto x - \frac{1}{v_1} e^{-\frac{1}{2}v_1} e^{v_1 x} \) is increasing for \( 0 \leq x \leq \frac{3}{2} \). Thus,

\[
\ln[1 + y(t^n)] \leq \frac{3}{2} - \frac{1}{v_1} \left[ 1 - e^{-\frac{1}{2}v_1} \right] \leq v_1 - \frac{1}{6} v_1^2,
\]

according to (2.19) on p.123 of [2].
Case 2: \(-\ln(1-v_i) < \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds \leq \frac{3}{2}\). Choose \(\tau \in (0, 1)\) such that
\[
\int_{t_{n-1}^*}^{t_n^*} r(s) \, ds = -\frac{\ln(1-v_i)}{v_1}.
\]
Then by (3.5) and (1.2),
\[
\ln(1 + y(t_n^*)) \\
\leq \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds + \int_{t_{n-1}^*}^{t_n^*} y(t) \left[ 1 - e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} \right] \, dt \\
\leq v_1 \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds + \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds - e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds \\
= v_1 \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds + \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds - e^{-\frac{1}{v_1} v_1 \left( 1 - \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds \right)} \left[ e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} - e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} \right] \\
= v_1 \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds + \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds - e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} \left[ 1 - e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} \right] \\
= v_1 \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds + \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds - e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} \left[ 1 - e^{-\frac{1}{v_1} \int_{t_{n-1}^*}^{t_n^*} r(s) \, ds} \right] \\
\leq \frac{3}{2} v_1 \left( 1 - v_1 \right) \ln(1 - v_1) - 1
\]
since \(x \mapsto v_1 x - e^{-v_1 (\frac{1}{2} - x)}\) is increasing for \(0 \leq x \leq \frac{3}{2}\). Thus, according to (2.21) on p.123 of [2],
\[
\ln(1 + y(t_n^*)) \leq v_1 - \frac{1}{6} v_1^2.
\]
Letting \(n \to \infty\) and \(\epsilon \to 0\), we have
\[
(3.6) \quad \ln(1 + u) \leq v - \frac{1}{6} v^2.
\]
Next, let \(\{s_n^*\}\) be an increasing sequence such that \(s_n^* \geq t_0 + 1\), \(y'(s_n^*) = 0\), \(\lim_{n \to \infty} y(s_n^*) = -v\) and \(\lim_{n \to \infty} s_n^* = \infty\) By (1.1), \(y(s_n^* - 1) = 0\). We will show that
\[
-\ln(1 + y(s_n^*)) \leq u_1 + \frac{1}{6} u_1^2.
\]
For \(t \in [s_n^* - 1, s_n^*]\), integrating (3.4) from \(t - 1\) to \(s_n^* - 1\), we have
\[
\ln(1 + y(t - 1)) \leq u_1 \int_{t - 1}^{s_n^* - 1} r(s) \, ds
\]
or
\[
y(t - 1) \leq -1 + e^{u_1 \int_{t - 1}^{s_n^* - 1} r(s) \, ds}.
\]
By (1.1)
\[
(3.7) \quad [\ln(1 + y(t))]' \geq -r(t) \left[ e^{u_1 \int_{t - 1}^{s_n^* - 1} r(s) \, ds} - 1 \right] \quad \text{for all } t \in [s_n^* - 1, s_n^*].
\]
Case 1: $\int_{s_n^{-1}}^{s_n^*} r(s) \, ds \leq 1$. Integrating (3.4) from $s_n^{-1}$ to $s_n^*$, we have

$$-\ln(1 + y(s_n^*)) \leq u_1 \int_{s_n^{-1}}^{s_n^*} r(s) \, ds \leq u_1 \leq u_1 + \frac{1}{6} u_1^2.$$  

Case 2: $1 < \int_{s_n^{-1}}^{s_n^*} r(s) \, ds \leq \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1}$. Clearly $u_1 > 2$ in this case. As in Case 1, we have

$$-\ln(1 + y(s_n^*)) \leq u_1 \int_{s_n^{-1}}^{s_n^*} r(s) \, ds \leq \frac{3}{2} u_1 - \ln(1 + u_1).$$

Claim: $\frac{1}{2} u_1 - \ln(1 + u_1) \leq u_1 + \frac{1}{6} u_1^2$. Indeed, if $u_1 \geq 3$, then this inequality holds trivially. And for $2 < u_1 < 3$,

$$\frac{1}{2} u_1 < \frac{3}{2} < \frac{3}{2} + \ln 3 < \frac{1}{6} u_1^2 + \ln(1 + u_1).$$

Case 3: $\frac{3}{2} - \frac{\ln(1 + u_1)}{u_1} < \int_{s_n^{-1}}^{s_n^*} r(s) \, ds \leq \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1}$. Choose $\tau \in (0, 1)$ such that $\int_{s_n^{-1}}^{s_n^*} r(s) \, ds = \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1}$. Then by (3.4) and (3.7), we have

$$-(\ln[1 + y(t)])' \leq \min \left\{ r(t) u_1, r(t) \left[ e^{u_1 \int_{s_n^{-1}}^{s_n^*} r(s) \, ds} - 1 \right] \right\}.$$ Consequently

$$-\ln(1 + y(s_n^*))$$

$$\leq \int_{s_n^{-1}}^{s_n^{-\tau}} r(t) u_1 \, dt + \int_{s_n^{-\tau}}^{s_n^*} r(t) \left[ e^{u_1 \int_{s_n^{-1}}^{s_n^{-\tau}} r(s) \, ds} - 1 \right] \, dt$$

$$\leq u_1 \left( \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1} \right) + e^{\tau u_1} \int_{s_n^{-\tau}}^{s_n^*} r(t) e^{-u_1 \int_{s_n^{-1}}^{s_n^{-\tau}} r(s) \, ds} - \int_{s_n^{-\tau}}^{s_n^*} r(t) \, dt$$

$$= u_1 \left( \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1} \right)$$

$$+ \frac{1}{u_1} e^{\tau u_1} \left[ \int_{s_n^{-1}}^{s_n^{-\tau}} r(s) \, ds - e^{-u_1 \int_{s_n^{-1}}^{s_n^{-\tau}} r(s) \, ds} - \int_{s_n^{-\tau}}^{s_n^*} r(t) \, dt \right]$$

$$= u_1 \left( \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1} \right)$$

$$+ \frac{1}{u_1} \left[ e^{u_1 \left( \frac{3}{2} - \int_{s_n^{-1}}^{s_n^{-\tau}} r(s) \, ds \right)} - e^{u_1 \left( \frac{3}{2} - \int_{s_n^{-1}}^{s_n^{-\tau}} r(s) \, ds \right)} - \int_{s_n^{-\tau}}^{s_n^*} r(t) \, dt \right]$$

$$\leq u_1 \left( \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1} \right)$$

$$+ \frac{1}{u_1} \left[ 1 + u_1 - 1 - u_1 \left( \frac{3}{2} - \int_{s_n^{-1}}^{s_n^{-\tau}} r(s) \, ds \right) \right] - \int_{s_n^{-\tau}}^{s_n^*} r(t) \, dt$$
due to the choice of $\tau$ and because $e^x \geq 1 + x$ for $x \geq 0$. Thus,
\begin{align*}
-\ln(1 + y(s_n)) & \leq u_1 \left( \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1} \right) + 1 - \frac{3}{2} \int_{s_n}^{s_n + \tau} r(s) \, ds - \int_{s_n - \tau}^{s_n} r(t) \, dt \\
& = u_1 \left( \frac{3}{2} - \frac{\ln(1 + u_1)}{u_1} \right) - \frac{1}{2} + \int_{s_n}^{s_n - \tau} r(s) \, ds \\
& \leq 1 - \ln(1 + u_1) + \frac{3}{2} u_1 - \frac{\ln(1 + u_1)}{u_1} \\
& = 1 - \frac{(1 + u_1)\ln(1 + u_1)}{u_1} + \frac{3}{2} u_1 \\
& \leq u_1 + \frac{1}{6} u_1^2
\end{align*}

by (2.22) on p.123 of [2].

Thus we have shown that
\[-\ln(1 + y(s_n)) \leq u_1 + \frac{1}{6} u_1^2.\]

Letting $n \to \infty$ and $\epsilon \to 0$, we have
\[-\ln(1 - v) \leq u + \frac{1}{6} u^2\]
or
\[1 - v \geq e^{-u - \frac{1}{6} u^2}.\]

Since $u, v$ satisfy the inequalities in (2.1) with $\alpha = \frac{1}{6}$, by Lemma 2.1 $u = v = 0$. This completes the proof.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA    T6G 2G1
E-mail address: joso@jazz.math.ualberta.ca

DEPARTMENT OF APPLIED MATHEMATICS, HUNAN UNIVERSITY, CHANGSHA, HUNAN 410082, PEOPLE'S REPUBLIC OF CHINA