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# GLOBAL ATTRACTIVITY FOR A POPULATION MODEL WITH TIME DELAY

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ABSTRACT. In this paper we give a sufficient condition which guarantees the global attractivity of the zero solution of a population growth equation.

## 1. INTRODUCTION

The delay differential equation

$$N'(t) = \beta N(t)\left[1 - \frac{N(t-q)}{K}\right], \qquad q > 0,$$

was proposed by Hutchinson [1] as a model for the growth of a single species. Using the change of variables $x(t) = \frac{N(qt)}{K} - 1$ and defining $r = \beta q$, the above "Hutchinson's equation" can be reduced to

$$x'(t) = -rx(t-1)[1 + x(t)]$$

(cf. Wright [4]).

More realistically, we may assume that the growth rate $r$ depends on time. Thus consider the delay differential equation

$$y'(t) = -r(t)[1 + y(t)]y(t-1), \qquad t \geq 0, \tag{1.1}$$

where $r : [0, \infty) \to (0, \infty)$ is continuous. This equation has been studied by Sugie [3] where it was shown that the zero solution of (1.1) is uniformly stable provided there exists a constant $0 < r_0 < \frac{3}{2}$ such that $r(t) \leq r_0$ for all $t \geq 0$.

In this paper, we are interested in the global attractivity of the zero solution. Due to the biological interpretation of (1.1), we are only interested in solutions $y(t)$ of (1.1) such that $y(t) \geq -1$ for $-1 \leq t \leq 0$ and $y(0) > -1$. In that case,

$$1 + y(t) = (1 + y(0))e^{-\int_0^t r(s)y(s-1)ds} > 0$$

and so $y(t) > -1$ for all $t \geq 0$.

We now state our main result.

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**Theorem 1.1.** *If*

$$\int_{t-1}^{t} r(s)\,ds \le \frac{3}{2} \qquad \text{for all large } t \tag{1.2}$$

*and*

$$\int_{0}^{\infty} r(s)\,ds = \infty, \tag{1.3}$$

*then every solution of* (1.1) *tends to zero as* $t \to \infty$.

In the case when $r(t)$ is a constant $r_0$ and $0 < r_0 \le \frac{3}{2}$, Theorem 1.1 was proved in Wright [4] (see also Kuang [2, Theorem 2.1, p.120]). It was conjectured in [4] (see also [2, Open Problem 4.1, p.171]) that the result is still true for $0 < r_0 \le \frac{\pi}{2}$. This question remains open. In [2], it was conjectured (Open Problem 4.2, p.171) that under the assumption $0 < r(t) < \frac{3}{2}$, every solution of (1.1) tends to zero. Theorem 1.1 gives a positive answer to this conjecture, provided (1.3) holds.

## 2. Preliminaries

**Lemma 2.1.** *Let* $0 < \alpha < \frac{1}{2}$. *The system of inequalities*

$$\begin{cases} u \le e^{v-\alpha v^2} - 1, \\ v \le 1 - e^{-u-\alpha u^2} \end{cases} \tag{2.1}$$

*has a unique solution*: $(v, u) = (0, 0)$ *in the nonnegative quadrant* $\{(v, u) : v \ge 0, u \ge 0\}$.

*Proof.* Assume that (2.1) has another solution in the first quadrant of the $v-u$ plane besides (0,0), say $(v_0, u_0)$. Then $u_0 > 0$ and $0 < v_0 < 1$. Define $\Gamma_1$ to be the curve: $u = e^{v-\alpha v^2} - 1$ and $\Gamma_2$ to be the curve: $v = 1 - e^{-u-\alpha u^2}$. Clearly $\frac{du}{dv}|_{(0,0)} = 1$, $\frac{d^2u}{dv^2}|_{(0,0)} = 1 - 2\alpha$ and $\frac{d^3u}{dv^3}|_{(0,0)} = 1 - 6\alpha$ for $\Gamma_1$. On the other hand, $\frac{du}{dv}|_{(0,0)} = 1$, $\frac{d^2u}{dv^2}|_{(0,0)} = 1 - 2\alpha$ and $\frac{d^3u}{dv^3} = 12\alpha^2 - 6\alpha + 2$ for $\Gamma_2$. Hence $\Gamma_2$ lies above $\Gamma_1$ near $(0, 0)$. The existence of $(v_0, u_0)$ implies that the curves $\Gamma_1$ and $\Gamma_2$ must intersect at a point in the first quadrant besides $(0, 0)$. Let $(v_1, u_1)$ be the first such point, i.e. $v_1$ is smallest. Then the slope of $\Gamma_1$ at $(u_1, v_1)$ is no less than the slope of $\Gamma_2$ at $(u_1, v_1)$, i.e.

$$(1 - 2\alpha v_1)e^{v_1 - \alpha v_1^2} \ge \frac{1}{1 + 2\alpha u_1} e^{u_1 + \alpha u_1^2}$$

or

$$(1 - 2\alpha v_1)(1 + 2\alpha u_1) \ge e^{u_1 - v_1 + \alpha(u_1^2 + v_1^2)}.$$

*Claim*: $u_1 > v_1$.

*Proof.* Let $\phi(x) = 1 - e^{-x-\alpha x^2} - x$. Then $\phi(0) = 0$ and $\phi'(x) < 0$ for $x > 0$, since $2\alpha < 1$. Thus $\phi(x) < 0$ for $x > 0$ and $v_1 = \phi(u_1) + u_1 < u_1$.

Using the inequality $e^x > 1 + x$ $(x > 0)$, we have

$$1 + 2\alpha(u_1 - v_1) - 4\alpha^2 u_1 v_1 > 1 + u_1 - v_1 + \alpha(u_1^2 + v_1^2)$$

or

$$(-1 + 2\alpha)(u_1 - v_1) - 4\alpha^2 u_1 v_1 > \alpha(u_1^2 + v_1^2)$$

which is a contradiction, since $0 < \alpha < \frac{1}{2}$. This completes the proof.

**Lemma 2.2.** *If* (1.3) *holds, then every nonoscillatory solution of* (1.1) *tends to zero as* $t \to \infty$.

*Proof.* Let $y(t)$ be an nonoscillatory solution of (1.1). Then there exists $t_0$ such that $y(t)$ is of one sign for $t \geq t_0$. Consider first the case $y(t) \leq 0$ for $t \geq t_0$. Since $1 + y(t) \geq 0$, by (1.1) $y'(t) \geq 0$. Thus $y(t)$ is increasing and $\lim_{t\to\infty} y(t) = -c$ exists. Clearly $-1 < y(t_0) \leq -c \leq 0$. Integrating (1.1), we get (for $t > t_0 + 1$)

$$-\ln(1 + y(t)) + \ln(1 + y(t_0 + 1)) = \int_{t_0+1}^{t} r(s)y(s-1)\,ds \leq -c\int_{t_0+1}^{t} r(s)\,ds.$$

By (1.3), the right-hand side tends to $-\infty$ as $t \to \infty$ unless $c = 0$. On the other hand, the left-hand side has a finite limit; therefore $c = 0$. Hence $\lim_{t\to\infty} y(t) = 0$. The case when $y(t)$ is eventually nonnegative is similar. This completes the proof.

The following lemma was essentially proved in [2, Theorem 3.1, p.128].

**Lemma 2.3.** *Assume* (1.2) *holds. Let* $y(t)$ *be an oscillatory solution of* (1.1). *Then* $y(t)$ *is bounded above and is bounded below from* $-1$ *for* $t \geq 0$.

*Proof.* Let $t_0 > 0$ be large enough so that (1.2) holds for all $t \geq t_0$. Let $t^*$ be a local maximum point of $y(t)$ $(t \geq t_0 + 1)$. Then $y'(t^*) = 0$ and by (1.1) $y(t^* - 1) = 0$. Integrating (1.1) from $t^* - 1$ to $t^*$, we have

$$1 + y(t^*) = e^{-\int_{t^*-1}^{t^*} r(s)y(s-1)\,ds}.$$

Since $y(s-1) \geq -1$, by (1.2)

$$1 + y(t^*) \leq e^{\int_{t^*-1}^{t^*} r(s)\,ds} \leq e^{\frac{3}{2}}$$

and $y(t^*) \leq e^{\frac{3}{2}} - 1$. Consequently, $\limsup_{t\to\infty} y(t) \leq e^{\frac{3}{2}} - 1$.

Next, let $t_*$ be a local minimum point of $y(t)$ $(t \geq t_0 + 3)$. Then $y'(t_*) = 0$ and $y(t_* - 1) = 0$. Proceeding similarly as before and using the fact that $y(s-1) \leq e^{\frac{3}{2}} - 1$, we have

$$1 + y(t_*) \geq e^{\int_{t_*-1}^{t_*} r(s)(1-e^{\frac{3}{2}})\,ds} = e^{-(e^{\frac{3}{2}}-1)\int_{t_*-1}^{t_*} r(s)\,ds} \geq e^{-(e^{\frac{3}{2}}-1)\frac{3}{2}}.$$

Hence,

$$y(t_*) \geq e^{-\frac{3(e^{\frac{3}{2}}-1)}{2}} - 1,$$

and $\liminf_{t\to\infty} y(t) \geq e^{-\frac{3}{2}(e^{\frac{3}{2}}-1)} - 1 > -1$.

## 3. Proof of Theorem 1.1

To complete the proof of Theorem 1.1, all we need is to show

**Lemma 3.1.** *If* (1.3) *holds, then every oscillatory solution of* (1.1) *tends to zero as* $t \to \infty$.

*Proof*. Let $y(t)$ be an oscillatory solution of (1.1). By Lemma 2.3 $y(t)$ $(t \geq 0)$ is bounded above and bounded below away from $-1$. Let

$$u = \limsup_{t\to\infty} y(t), \qquad -v = \liminf_{t\to\infty} y(t). \tag{3.1}$$

Then $0 \leq v < 1$ and $0 \leq u < \infty$. It suffices to show that $u = v = 0$. For any $\epsilon$, choose $t_0 = t_0(\epsilon)$ such that

$$-v_1 \equiv -v - \epsilon < y(t-1) < u + \epsilon \equiv u_1, \qquad \text{for } t \geq t_0. \tag{3.2}$$

We assume that $\epsilon$ is small enough so that $0 < v_1 < 1$ and that $t_0$ is large enough so that (1.2) holds for $t \geq t_0 - 2$. Using (1.1), we have

$$y'(t) \leq r(t)[1 + y(t)]v_1, \qquad t \geq t_0, \tag{3.3}$$

and

$$y'(t) \geq -r(t)[1 + y(t)]u_1, \qquad t \geq t_0. \tag{3.4}$$

Let $\{t_n^*\}$ be an increasing sequence such that $t_n^* \geq t_0 + 1$, $y'(t_n^*) = 0$, $\lim_{n\to\infty} t_n^* = \infty$ and $\lim_{n\to\infty} y(t_n^*) = u$. By (1.1), $y(t_n^* - 1) = 0$. For $t \in [t_n^* - 1, t_n^*]$, we can integrate (3.3) from $t-1$ to $t_n^* - 1$ and get

$$-\ln[1 + y(t-1)] \leq v_1 \int_{t-1}^{t_n^*-1} r(s)\,ds$$

or

$$y(t-1) \geq -1 + e^{-v_1 \int_{t-1}^{t_n^*-1} r(s)\,ds} \qquad \text{for } t \in [t_n^* - 1, t_n^*].$$

By (1.1) it follows that

$$y'(t) \leq r(t)[1 + y(t)]\left[1 - e^{-v_1 \int_{t-1}^{t_n^*-1} r(s)\,ds}\right], \qquad t \in [t_n^* - 1, t_n^*].$$

Combining this with (3.3), we have

$$(\ln[1 + y(t)])' \leq \min\left\{r(t)v_1, r(t)\left[1 - e^{-v_1 \int_{t-1}^{t_n^*-1} r(s)\,ds}\right]\right\}, \qquad t \in [t_n^* - 1, t_n^*]. \tag{3.5}$$

We will prove that

$$\ln[1 + y(t_n^*)] \leq v_1 - \frac{1}{6}v_1^2.$$

There are two possibilities.

*Case* 1: $\int_{t_n^*-1}^{t_n^*} r(s)\,ds \le -\frac{\ln(1-v_1)}{v_1}$. Then by (3.5)

$$\begin{aligned}
\ln[1+y(t_n^*)] &\le \int_{t_n^*-1}^{t_n^*} r(t)\left[1-e^{-v_1\int_{t-1}^{t_n^*-1} r(s)\,ds}\right]dt \\
&= \int_{t_n^*-1}^{t_n^*} r(t)\left[1-e^{-v_1\int_{t-1}^{t} r(s)\,ds+v_1\int_{t_n^*-1}^{t} r(s)\,ds}\right]dt \\
&\le \int_{t_n^*-1}^{t_n^*} r(t)\left[1-e^{-\frac{3}{2}v_1}e^{v_1\int_{t_n^*-1}^{t} r(s)\,ds}\right]dt \\
&= \int_{t_n^*-1}^{t_n^*} r(t)\,dt-e^{-\frac{3}{2}v_1}\int_{t_n^*-1}^{t_n^*} r(t)e^{v_1\int_{t_n^*-1}^{t} r(s)\,ds}\,dt \\
&= \int_{t_n^*-1}^{t_n^*} r(t)\,dt-e^{-\frac{3}{2}v_1}\frac{1}{v_1}\left[e^{v_1\int_{t_n^*-1}^{t_n^*} r(s)\,ds}-1\right] \\
&= \int_{t_n^*-1}^{t_n^*} r(t)\,dt-\frac{1}{v_1}e^{-v_1\left(\frac{3}{2}-\int_{t_n^*-1}^{t_n^*} r(s)\,ds\right)}\left[1-e^{-v_1\int_{t_n^*-1}^{t_n^*} r(s)\,ds}\right].
\end{aligned}$$

The function $\phi(x)=x-\frac{1}{v_1}e^{-v_1(\frac{3}{2}-x)}(1-e^{-v_1x})$ is increasing for $0\le x\le\frac{3}{2}$. Thus for $\int_{t_n^*-1}^{t_n^*} r(t)\,dt\le-\frac{\ln(1-v_1)}{v_1}\le\frac{3}{2}$, we have

$$\begin{aligned}
\ln[1+y(t_n^*)] &\le -\frac{\ln(1-v_1)}{v_1}-\frac{1}{v_1}e^{-v_1\left(\frac{3}{2}+\frac{\ln(1-v_1)}{v_1}\right)}[1-e^{\ln(1-v_1)}] \\
&= -\frac{\ln(1-v_1)}{v_1}-e^{-v_1\left(\frac{3}{2}+\frac{\ln(1-v_1)}{v_1}\right)}.
\end{aligned}$$

Using the fact $e^{-x}>1-x$ for $x>0$, we have

$$\ln[1+y(t_n^*)]\le -1+\frac{3}{2}v_1-\frac{(1-v_1)\ln(1-v_1)}{v_1}<v_1-\frac{1}{6}v_1^2,$$

according to (2.21) on p.123 of [2].

For $\int_{t_n^*-1}^{t_n^*} r(t)\,dt\le\frac{3}{2}<-\frac{\ln(1-v_1)}{v_1}$, we have

$$\ln[1+y(t_n^*)]\le\int_{t_n^*-1}^{t_n^*} r(t)\,dt-\frac{1}{v_1}\left[e^{-\frac{3}{2}v_1}e^{v_1\int_{t_n^*-1}^{t_n^*} r(t)\,dt}-e^{-\frac{3}{2}v_1}\right].$$

The function $x\mapsto x-\frac{1}{v_1}e^{-\frac{3}{2}v_1}e^{v_1x}$ is increasing for $0\le x\le\frac{3}{2}$. Thus,

$$\ln[1+y(t_n^*)]\le\frac{3}{2}-\frac{1}{v_1}\left[1-e^{-\frac{3}{2}v_1}\right]\le v_1-\frac{1}{6}v_1^2$$

according to (2.19) on p.123 of [2].

*Case* 2: $-\frac{\ln(1-v_1)}{v_1} < \int_{t_n^*-1}^{t_n^*} r(s)\,ds \le \frac{3}{2}$. Choose $\tau \in (0, 1)$ such that

$$\int_{t_n^*-\tau}^{t_n^*} r(s)\,ds = -\frac{\ln(1-v_1)}{v_1}.$$

Then by (3.5) and (1.2),

$$\begin{aligned}
&\ln(1+y(t_n^*))\\
&\le \int_{t_n^*-1}^{t_n^*-\tau} r(s)v_1\,ds + \int_{t_n^*-\tau}^{t_n^*} r(t)\left[1-e^{-v_1\int_{t-1}^{t_n^*-1} r(s)\,ds}\right]dt\\
&\le v_1\int_{t_n^*-1}^{t_n^*-\tau} r(s)\,ds + \int_{t_n^*-\tau}^{t_n^*} r(s)\,ds - e^{-\frac{3}{2}v_1}\int_{t_n^*-\tau}^{t_n^*} r(t)e^{v_1\int_{t_n^*-1}^{t} r(s)\,ds}\,dt\\
&= v_1\int_{t_n^*-1}^{t_n^*-\tau} r(s)\,ds + \int_{t_n^*-\tau}^{t_n^*} r(s)\,ds - \frac{1}{v_1}e^{-\frac{3}{2}v_1}\left[e^{v_1\int_{t_n^*-1}^{t_n^*} r(s)\,ds} - e^{v_1\int_{t_n^*-1}^{t_n^*-\tau} r(s)\,ds}\right]\\
&= v_1\int_{t_n^*-1}^{t_n^*-\tau} r(s)\,ds + \int_{t_n^*-\tau}^{t_n^*} r(s)\,ds - \frac{1}{v_1}e^{-v_1\left(\frac{3}{2}-\int_{t_n^*-1}^{t_n^*} r(s)\,ds\right)}\left[1-e^{-v_1\int_{t_n^*-\tau}^{t_n^*} r(s)\,ds}\right]\\
&= v_1\int_{t_n^*-1}^{t_n^*-\tau} r(s)\,ds + \int_{t_n^*-\tau}^{t_n^*} r(s)\,ds - e^{-v_1\left(\frac{3}{2}-\int_{t_n^*-1}^{t_n^*} r(s)\,ds\right)}\\
&= v_1\int_{t_n^*-1}^{t_n^*} r(s)\,ds + (1-v_1)\int_{t_n^*-\tau}^{t_n^*} r(s)\,ds - e^{-v_1\left(\frac{3}{2}-\int_{t_n^*-1}^{t_n^*} r(s)\,ds\right)}\\
&\le \frac{3}{2}v_1 - \frac{(1-v_1)\ln(1-v_1)}{v_1} - 1
\end{aligned}$$

since $x \mapsto v_1x - e^{-v_1(\frac{3}{2}-x)}$ is increasing for $0 \le x \le \frac{3}{2}$. Thus, according to (2.21) on p.123 of [2],

$$\ln(1+y(t_n^*)) \le v_1 - \frac{1}{6}v_1^2.$$

Letting $n \to \infty$ and $\epsilon \to 0$, we have

$$\ln(1+u) \le v - \frac{1}{6}v^2. \tag{3.6}$$

Next, let $\{s_n^*\}$ be an increasing sequence such that $s_n^* \ge t_0 + 1$, $y'(s_n^*) = 0$, $\lim_{n\to\infty} y(s_n^*) = -v$ and $\lim_{n\to\infty} s_n^* = \infty$ By (1.1), $y(s_n^*-1) = 0$. We will show that

$$-\ln(1+y(s_n^*)) \le u_1 + \frac{1}{6}u_1^2.$$

For $t \in [s_n^*-1, s_n^*]$, integrating (3.4) from $t-1$ to $s_n^*-1$, we have

$$\ln(1+y(t-1)) \le u_1\int_{t-1}^{s_n^*-1} r(s)\,ds$$

or

$$y(t-1) \le -1 + e^{u_1\int_{t-1}^{s_n^*-1} r(s)\,ds}.$$

By (1.1)

$$[\ln(1+y(t))]' \ge -r(t)\left[e^{u_1\int_{t-1}^{s_n^*-1} r(s)\,ds} - 1\right] \qquad \text{for all } t \in [s_n^*-1, s_n^*]. \tag{3.7}$$

*Case* 1: $\int_{s_n^*-1}^{s_n^*} r(s)\,ds \le 1$. Integrating (3.4) from $s_n^*-1$ to $s_n^*$, we have

$$-\ln(1+y(s_n^*)) \le u_1 \int_{s_n^*-1}^{s_n^*} r(s)\,ds \le u_1 \le u_1 + \frac{1}{6}u_1^2.$$

*Case* 2: $1 < \int_{s_n^*-1}^{s_n^*} r(s)\,ds \le \frac{3}{2} - \frac{\ln(1+u_1)}{u_1}$. Clearly $u_1 > 2$ in this case. As in Case 1, we have

$$-\ln(1+y(s_n^*)) \le u_1 \int_{s_n^*-1}^{s_n^*} r(s)\,ds \le \frac{3}{2}u_1 - \ln(1+u_1).$$

*Claim*: $\frac{3}{2}u_1 - \ln(1+u_1) \le u_1 + \frac{1}{6}u_1^2$. Indeed, if $u_1 \ge 3$, then this inequality holds trivially. And for $2 < u_1 < 3$,

$$\frac{1}{2}u_1 < \frac{3}{2} < \frac{2}{3} + \ln 3 < \frac{1}{6}u_1^2 + \ln(1+u_1).$$

*Case* 3: $\frac{3}{2} - \frac{\ln(1+u_1)}{u_1} < \int_{s_n^*-1}^{s_n^*} r(s)\,ds \le \frac{3}{2}$. Choose $\tau \in (0, 1)$ such that $\int_{s_n^*-1}^{s_n^*-\tau} r(s)\,ds = \frac{3}{2} - \frac{\ln(1+u_1)}{u_1}$. Then by (3.4) and (3.7), we have

$$-(\ln[1+y(t)]') \le \min\left\{ r(t)u_1\,,\; r(t)\left[e^{u_1 \int_{t-1}^{s_n^*-1} r(s)\,ds} - 1\right]\right\}.$$

Consequently

$$\begin{aligned}
&-\ln(1+y(s_n^*)) \\
&\quad\le \int_{s_n^*-1}^{s_n^*-\tau} r(t)u_1\,dt + \int_{s_n^*-\tau}^{s_n^*} r(t)\left[e^{u_1\int_{t-1}^{s_n^*-1} r(s)\,ds} - 1\right] dt \\
&\quad\le u_1\left(\frac{3}{2} - \frac{\ln(1+u_1)}{u_1}\right) + e^{\frac{3}{2}u_1}\int_{s_n^*-\tau}^{s_n^*} r(t)e^{-u_1\int_{s_n^*-1}^{t} r(s)\,ds} - \int_{s_n^*-\tau}^{s_n^*} r(t)dt \\
&\quad= u_1\left(\frac{3}{2} - \frac{\ln(1+u_1)}{u_1}\right) \\
&\qquad+ \frac{1}{u_1}e^{\frac{3}{2}u_1}\left[e^{-u_1\int_{s_n^*-1}^{s_n^*-\tau} r(s)\,ds} - e^{-u_1\int_{s_n^*-1}^{s_n^*} r(s)\,ds}\right] - \int_{s_n^*-\tau}^{s_n^*} r(t)dt \\
&\quad= u_1\left(\frac{3}{2} - \frac{\ln(1+u_1)}{u_1}\right) \\
&\qquad+ \frac{1}{u_1}\left[e^{u_1\left(\frac{3}{2}-\int_{s_n^*-1}^{s_n^*-\tau} r(s)\,ds\right)} - e^{u_1\left(\frac{3}{2}-\int_{s_n^*-1}^{s_n^*} r(s)\,ds\right)}\right] - \int_{s_n^*-\tau}^{s_n^*} r(t)dt \\
&\quad\le u_1\left(\frac{3}{2} - \frac{\ln(1+u_1)}{u_1}\right) \\
&\qquad+ \frac{1}{u_1}\left[1 + u_1 - 1 - u_1\left(\frac{3}{2} - \int_{s_n^*-1}^{s_n^*} r(s)\,ds\right)\right] - \int_{s_n^*-\tau}^{s_n^*} r(t)dt
\end{aligned}$$

due to the choice of $\tau$ and because $e^x \geq 1 + x$ for $x \geq 0$. Thus,

$$\begin{aligned}
&-\ln(1 + y(s_n^*)) \\
&\leq u_1\left(\frac{3}{2} - \frac{\ln(1+u_1)}{u_1}\right) + 1 - \frac{3}{2} + \int_{s_n^*-1}^{s_n^*} r(s)\,ds - \int_{s_n^*-\tau}^{s_n^*} r(t)dt \\
&= u_1\left(\frac{3}{2} - \frac{\ln(1+u_1)}{u_1}\right) - \frac{1}{2} + \int_{s_n^*-1}^{s_n^*-\tau} r(s)\,ds \\
&\leq 1 - \ln(1+u_1) + \frac{3}{2}u_1 - \frac{\ln(1+u_1)}{u_1} \\
&= 1 - \frac{(1+u_1)\ln(1+u_1)}{u_1} + \frac{3}{2}u_1 \\
&\leq u_1 + \frac{1}{6}u_1^2
\end{aligned}$$

by (2.22) on p.123 of [2].

Thus we have shown that

$$-\ln(1 + y(s_n^*)) \leq u_1 + \frac{1}{6}u_1^2.$$

Letting $n \to \infty$ and $\epsilon \to 0$, we have

$$-\ln(1 - v) \leq u + \frac{1}{6}u^2$$

or

$$1 - v \geq e^{-u-\frac{1}{6}u^2}.$$

Since $u, v$ satisfy the inequalities in (2.1) with $\alpha = \frac{1}{6}$, by Lemma 2.1 $u = v = 0$. This completes the proof.

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