PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 9, September 1995

A NOTE ON COHOMOLOGICAL DIMENSION OF APPROXIMATE MOVABLE SPACES

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(Communicated by James E. West)

ABSTRACT. We show that any approximate movable compact metric space X satisfies the equality $\dim X = \dim_{\mathbb{Z}} X$ without finite dimensional condition. Thus there is no approximate movable compact metric space X with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X < \infty$. Since ANRs and some generalized ANRs are approximate movable, they satisfy the above equality.

All spaces are compact metric and all polyhedra are finite. Let X be a space. By $\dim X$ and $\dim_{\mathbb{Z}} X$ we denote covering dimension and integral cohomological dimension of X, respectively. It is well known (the fundamental cohomological dimension theorem) that if $\dim X$ is finite, then $\dim X = \dim_{\mathbb{Z}} X$ (see P. S. Aleksandrov [1]). Recently, A. N. Dranishnikov [5] constructed a space X with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X = 3$. So his example means that the equality $\dim X = \dim_{\mathbb{Z}} X$ does not hold without finite dimensional condition. In this note we investigate this equality for some nice spaces:

Theorem 1. *If X is approximate movable, then $\dim X = \dim_{\mathbb{Z}} X$ holds.*

Corollary 2. *There does not exist an approximate movable space X with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X < \infty$.*

In [9] the author introduced an approximate shape theory and approximate movability which is an approximate invariant property.

Let X be a space, and let $\mathscr{X} = \{P_i, f_{ij}, \mathbb{N}\}$ be an inverse sequence of polyhedra P_i and maps $f_{ji}\colon X_j \to X_i$, $i < j$, such that X is an inverse limit of $\mathscr{X}$. Lemma (1.6) of [9, II] means the following:

Lemma 3. *X is approximate movable if and only if for each integer k and each $\varepsilon > 0$ there is an integer $j > k$ with the following property: For each integer $i \geq k$ there is a map $r_i\colon X_j \to X_i$ such that $f_{ik} r_i$ and f_{jk} are ε-near.*

For our proof we need some characterizations of dimension and cohomological dimension. For any integer n and any triangulation K, $K^{(n)}$ denotes the n-th skeleton of K and $|K|$ denotes the realization of K. Lemmas 4 and 5 are Theorem 4.1 and Theorem 5.1 of [8].

Received by the editors November 15, 1993.
1991 *Mathematics Subject Classification.* Primary 54C55, 54C56, 54F45.
Key words and phrases. Covering dimension, cohomological dimension, ANR, generalized ANR, approximate movability.

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Lemma 4. *X has $\dim X \leq n$ if and only if for each integer k and each $\varepsilon > 0$ there exist an integer $j > k$, a triangulation L_k of P_k, and a map $g_{jk}\colon P_j \to |L_k^{(n)}|$ which is ε-close to f_{jk}.*

Lemma 5 (R. D. Edwards). *X has $\dim_{\mathbb{Z}} X \leq n$ if and only if, given an integer $i \geq 1$, for each integer k and each $\varepsilon > 0$ there is a triangulation L_k of P_k and an integer $j > k$ such that for any triangulation L_j of P_j there is a map $g_{jk}\colon |L_j^{(n+i)}| \to |L_k^{(n)}|$ which is ε-close to the restriction of f_{jk}.*

Proof of Theorem 1. First, we show the inequality $\dim X \leq \dim_{\mathbb{Z}} X$. If $\dim_{\mathbb{Z}} X = \infty$, there is nothing to prove, so we consider the case $\dim_{\mathbb{Z}} X \leq n < \infty$ for some integer n. Take any integer k and any $\varepsilon > 0$. Put $\delta = \varepsilon/3$. Since X is approximate movable, by Lemma 3 there is an integer $j \geq k$ satisfying

(1) for each $i \geq k$ there is a map $r_i\colon P_j \to P_i$ such that $f_{ik} r_i$ and f_{jk} are δ-near.

Since P_j is a finite polyhedron, take a triangulation L_j of P_j and let $s = \dim L_j < \infty$. Since $\dim_{\mathbb{Z}} X \leq n < \infty$, by Lemma 5 there exist a triangulation L_k of P_k and an integer $i > k$ such that

(2) for any triangulation L_i of P_i there is a map $g_{ik}\colon |L_i^{(n+s)}| \to |L_k^{(n)}|$ which is δ-close to the restriction of f_{ik}.

Since $f_{ik}\colon X_i \to X_k$ is uniform, there is an $\eta > 0$ such that if points x and x' in X_i are η-near, then $f_{ik}(x)$ and $f_{ik}(x')$ are δ-near. Take a triangulation L_i of P_i such that any simplex of L_i has a diameter $< \eta/2$. By the simplicial approximation theorem there are a subdivision L_j' of L_j and a simplicial map $\varphi\colon L_j' \to L_i$ which approximates r_i, i.e., its realization $|\varphi|$ and r_i are η-near. By the choice of η, $f_{ik}|\varphi|$ and $f_{ik} r_i$ are δ-near. Since φ is simplicial and $s = \dim L_j = \dim L_j'$, φ induces a map $h = |\varphi|\colon P_j = |L_j'| = |L_j'^{(s)}| \to |L_i^{(s)}| \subset |L_i^{(n+s)}|$. Thus

(3) $f_{ik} h$ and $f_{ik} r_i$ are δ-near.

Since $h\colon P_j \to |L_i^{(n+s)}|$, by (2)

(4) $g_{ik} h$ and $f_{ik} h$ are δ-near.

By (1), (3) and (4), f_{jk} and $g_{ik} h\colon P_j \to |L_k^{(n)}|$ are ε-near. Thus j and the map $g_{ik} h$ satisfies the condition in Lemma 4 for k and ε. Then $\dim X \leq n$. This means the inequality $\dim X \leq \dim_{\mathbb{Z}} X$.

Next, we show the inequality $\dim_{\mathbb{Z}} X \leq \dim X$. If $\dim X = \infty$, there is nothing to prove, so we consider the case $\dim X \leq n < \infty$ for some integer n. It is easy to show $\dim_{\mathbb{Z}} X \leq n$ by Lemmas 4 and 5. This means the inequality $\dim_{\mathbb{Z}} X \leq \dim X$. Therefore, we have the required equality.

Corollary 2 follows from Theorem 1 and also means that Dranishnikov's example is not approximate movable.

Borsuk [2] introduced an absolute neighborhood retract, in notation ANR. There are many generalizations of ANR. Noguchi [7] introduced an absolute neighborhood retract in the sense of Noguchi, in notation $\mathrm{AANR_N}$. Clapp [4] introduced an absolute neighborhood retract in the sense of Clapp, in notation $\mathrm{AANR_C}$. Borsuk [3] introduced a nearly extendable set, in notation NE-set. Mardešić [6] introduced an approximate polyhedron, in notation AP. In [9, II, III] we gave their descriptions in approximate shape theory. By (II.4.7), (II.4.6), (II.2.18), (II.2.3), (II.5.10), (II.5.11) and (III.1.2) of [9], these ANR,

AANR_{N}, AANR_{C}, NE-set and AP are approximate movable (see the table of [9, II, p. 337]). Thus we have

Corollary 6. *If X is ANR, ANR_N, ANR_C, NE-set or AP, then* $\dim X = \dim_{\mathbb{Z}} X$ *holds.*

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