A NOTE ON COHOMOLOGICAL DIMENSION
OF APPROXIMATE MOVABLE SPACES

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Abstract. We show that any approximate movable compact metric space $X$ satisfies the equality $\dim X = \dim_2 X$ without finite dimensional condition. Thus there is no approximate movable compact metric space $X$ with $\dim X = \infty$ and $\dim_2 X < \infty$. Since ANRs and some generalized ANRs are approximate movable, they satisfy the above equality.

All spaces are compact metric and all polyhedra are finite. Let $X$ be a space. By $\dim X$ and $\dim_2 X$ we denote covering dimension and integral cohomological dimension of $X$, respectively. It is well known (the fundamental cohomological dimension theorem) that if $\dim X$ is finite, then $\dim X = \dim_2 X$ (see P. S. Aleksandrov [1]). Recently, A. N. Dranishnikov [5] constructed a space $X$ with $\dim X = \infty$ and $\dim_2 X = 3$. So his example means that the equality $\dim X = \dim_2 X$ does not hold without finite dimensional condition. In this note we investigate this equality for some nice spaces:

Theorem 1. If $X$ is approximate movable, then $\dim X = \dim_2 X$ holds.

Corollary 2. There does not exist an approximate movable space $X$ with $\dim X = \infty$ and $\dim_2 X < \infty$.

In [9] the author introduced an approximate shape theory and approximate movability which is an approximate invariant property.

Let $X$ be a space, and let $\mathcal{H} = \{P_i, f_{ij}, N\}$ be an inverse sequence of polyhedra $P_i$ and maps $f_{ij}: X_j \rightarrow X_i$, $i < j$, such that $X$ is an inverse limit of $\mathcal{H}$. Lemma (1.6) of [9, II] means the following:

Lemma 3. $X$ is approximate movable if and only if for each integer $k$ and each $\varepsilon > 0$ there is an integer $j > k$ with the following property: For each integer $i \geq k$ there is a map $r_i: X_j \rightarrow X_i$ such that $f_{ik}r_i$ and $f_{jk}$ are $\varepsilon$-near.

For our proof we need some characterizations of dimension and cohomological dimension. For any integer $n$ and any triangulation $K$, $K^{(n)}$ denotes the $n$-th skeleton of $K$ and $|K|$ denotes the realization of $K$. Lemmas 4 and 5 are Theorem 4.1 and Theorem 5.1 of [8].

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Lemma 4. \( X \) has \( \dim X \leq n \) if and only if for each integer \( k \) and each \( \varepsilon > 0 \) there exist an integer \( j > k \), a triangulation \( L_k \) of \( P_k \), and a map \( g_{jk}: P_j \to |L_k(n)| \) which is \( \varepsilon \)-close to \( f_{jk} \).

Lemma 5 (R. D. Edwards). \( X \) has \( \dim_Z X \leq n \) if and only if, given an integer \( i \geq 1 \), for each integer \( k \) and each \( \varepsilon > 0 \) there is a triangulation \( L_k \) of \( P_k \) and an integer \( j > k \) such that for any triangulation \( L_j \) of \( P_j \) there is a map \( g_{jk}: |L_j(n+i)| \to |L_k(n)| \) which is \( \varepsilon \)-close to the restriction of \( f_{jk} \).

Proof of Theorem 1. First, we show the inequality \( \dim X \leq \dim_Z X \). If \( \dim_Z X = \infty \), there is nothing to prove, so we consider the case \( \dim_Z X \leq n < \infty \) for some integer \( n \). Take any integer \( k \) and any \( \varepsilon > 0 \). Put \( \delta = \varepsilon / 3 \). Since \( X \) is approximate movable, by Lemma 3 there is an integer \( j > k \) satisfying

(1) for each \( i \geq k \) there is a map \( r_i: P_j \to P_i \) such that \( f_{ik}r_i \) and \( f_{jk} \) are \( \delta \)-near.

Since \( P_j \) is a finite polyhedron, take a triangulation \( L_j \) of \( P_j \) and let \( s = \dim L_j < \infty \). Since \( \dim_Z X \leq n < \infty \), by Lemma 5 there exist a triangulation \( L_k \) of \( P_k \) and an integer \( i > k \) such that

(2) for any triangulation \( L_i \) of \( P_i \) there is a map \( g_{ik}: |L_i(n+i)| \to |L_k(n)| \) which is \( \delta \)-close to the restriction of \( f_{ik} \).

Since \( f_{ik}: X_i \to X_k \) is uniform, there is an \( \eta > 0 \) such that if points \( x \) and \( x' \) in \( X_i \) are \( \eta \)-near, then \( f_{ik}(x) \) and \( f_{ik}(x') \) are \( \delta \)-near. Take a triangulation \( L_i \) of \( P_i \) such that any simplex of \( L_i \) has a diameter \( < \eta / 2 \). By the simplicial approximation theorem there are a subdivision \( L_j' \) of \( L_j \) and a simplicial map \( \varphi: L_j' \to L_i \) which approximates \( r_i \), i.e., its realization \( |\varphi| \) and \( r_i \) are \( \eta \)-near. By the choice of \( \eta \), \( f_{ik}|\varphi| \) and \( f_{ik}r_i \) are \( \delta \)-near. Since \( \varphi \) is simplicial and \( s = \dim L_j = \dim L_j' \), \( \varphi \) induces a map \( h = |\varphi|: P_j \to |L_j'| = |L_j'(s)| \to |L_i(s)| \subset |L_i(n+i)| \). Thus

(3) \( f_{ik}h \) and \( f_{ik}r_i \) are \( \delta \)-near.

Since \( h: P_j \to |L_j(n+i)| \), by (2)

(4) \( g_{ik}h \) and \( f_{ik}h \) are \( \delta \)-near.

By (1), (3) and (4), \( f_{jk} \) and \( g_{ik}h \): \( P_j \to |L_k(n)| \) are \( \varepsilon \)-near. Thus \( j \) and the map \( g_{ik}h \) satisfies the condition in Lemma 4 for \( k \) and \( \varepsilon \). Then \( \dim X \leq n \). This means the inequality \( \dim X \leq \dim_Z X \).

Next, we show the inequality \( \dim_Z X \leq \dim X \). If \( \dim X = \infty \), there is nothing to prove, so we consider the case \( \dim X \leq n < \infty \) for some integer \( n \). It is easy to show \( \dim_Z X \leq n \) by Lemmas 4 and 5. This means the inequality \( \dim_Z X \leq \dim X \). Therefore, we have the required equality.

Corollary 2 follows from Theorem 1 and also means that Dranishnikov’s example is not approximate movable.

Borsuk [2] introduced an absolute neighborhood retract, in notation ANR. There are many generalizations of ANR. Noguchi [7] introduced an absolute neighborhood retract in the sense of Noguchi, in notation AANR_N. Clapp [4] introduced an absolute neighborhood retract in the sense of Clapp, in notation AANR.C. Borsuk [3] introduced a nearly extendable set, in notation NE-set. Mardešić [6] introduced an approximate polyhedron, in notation AP. In [9, II, III] we gave their descriptions in approximate shape theory. By (II.4.7), (II.4.6), (II.2.18), (II.2.3), (II.5.10), (II.5.11) and (III.1.2) of [9], these ANR,
AANR\textsubscript{N}, AANR\textsubscript{C}, NE-set and AP are approximate movable (see the table of [9, II, p. 337]). Thus we have

**Corollary 6.** If \( X \) is ANR, ANRN, ANRC, NE-set or AP, then \( \dim X = \dim \mathbb{Z} X \) holds.

**References**


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