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ADMISSIBILITY, THE LOCALLY CONVEX APPROXIMATION PROPERTY, AND THE AR-PROPERTY IN LINEAR METRIC SPACES

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ABSTRACT. We introduce the notion of the locally convex approximation property (the LCAP) for convex sets in linear metric spaces. The LCAP is an extension of the notion of admissibility of Klee. We prove that any convex set with the LCAP is an AR.

1. INTRODUCTION

The following question (we call it *the AR-problem*) is among the most outstanding problems in infinite-dimensional topology:

1.1. **Problem.** Is every convex set in a linear metric space an AR? See [G, W, Problems LS1, LS6].

For locally convex spaces Problem 1.1 was settled affirmatively by Dugundji [D]. However, this problem remains open for nonlocally convex linear metric spaces and is one of the most resistant open problems in infinite-dimensional topology.

In this paper we provide some partial answers to the AR-problem. Our idea of attacking Problem 1.1 is to approximate convex sets in linear metric spaces by convex sets in locally convex spaces. We introduce the notion of the *locally convex approximation property* (the *LCAP*) for convex sets in linear metric spaces. The LCAP is an extension of the notion of admissibility of Klee [K1, K2] but has some advantages: In fact, we prove that the LCAP does imply the AR-property for all convex sets (not necessarily separable); meanwhile admissibility does not have this property: van der Bijl and van Mill [BM] have shown that even in the separable case admissibility does not imply the AR-property for convex sets unless all linear metric spaces are AR.

Roughly speaking, our theorem states that if a convex set X can be "approximated", in some sense, by convex subsets in locally convex spaces, then X is an AR. For instance, any convex set which is a union of an increasing sequence of convex subsets each of which can be affinely imbedded into a locally convex space, is an AR.

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Our results in this paper provide new examples of convex sets with the AR-property and raise some new problems for further investigation to the AR-problem in linear metric spaces, one of the most difficult problems in infinite-dimensional topology.

Notation and Conventions. In this paper all maps are assumed to be continuous. By a linear metric space we mean a topological linear space X which is metrizable. We assume that X is equipped with an F-norm $\|\cdot\|$ such that (see [Re])

(1) $\|\lambda x\| \leq \|x\|$ for every $x \in X$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

The zero element of X is denoted by θ. A locally convex space is a linear metric space which possesses a basis of neighbourhoods of θ consisting of convex sets.

Let E be a subset of a linear space X. By conv E we denote the convex hull of E in X, and span E denotes the linear subspace of X spanned by E. For $x \in X$, we write:

$$\|x - E\| = \inf\{\|x - y\| : y \in E\}.$$

For undefined notation, see [Bo, BP, Re].

2. The locally convex approximation property and admissible convex sets

Following Klee [K1, K2] a convex set X is *admissible* iff for every compact subset A of X and for every $\varepsilon > 0$ there exists a map f from A into a finite-dimensional subset of X such that $\|x - f(x)\| < \varepsilon$ for every $x \in A$.

The notion of admissibility is quite useful for detecting the AR-property for compact convex sets in linear metric spaces. In fact, Klee [K1, K2] proved that every admissible convex set X has the *compact extension property*, that is, any map into X defined on a compact subset of a metric space extends to the whole space. In particular, any admissible compact convex set is an AR. Using this property, we have discovered the AR-property for certain Roberts spaces [R1, R2]; see [NT1]. (By a *Roberts space* we mean any compact convex set with no extreme points constructed by Roberts' method of needle point spaces; see [NT1, NT2, N2, NST] for some results about the AR-property and the fixed point property for Roberts spaces.) Unfortunately, as shown in [BM], admissibility *does not* bring any information about the AR-property for noncompact convex sets. Our feeling is that the notion of admissibility should improve in order that we could have a better range of applications, especially to the noncompact case. The notion of the LCAP introduced in this section comes from this context.

2.1. **Definition.** A convex set X in a linear metric space is *LC-convex* iff for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, X) > 0$ such that for every finite subset $A \subset X$ with $\operatorname{diam} A < \delta$ we have $\operatorname{diam} \operatorname{conv} A < \varepsilon$.

Clearly, any convex set in a locally convex linear metric space is LC-convex.

We have the following obvious observations:

2.2. **Fact.** If E is a LC-convex subset of a linear metric space X, then the closure $\overline{E}$ of E in X is also LC-convex.

From the Dugundji theorem (see [D, BP]), we get

2.3. **Fact.** If E is an LC-convex closed subset of a linear metric space X, then there is a Dugundji retraction $r: X \to E$ such that

$$\|r(x) - x\| < 3\varepsilon \quad \text{whenever } x \in X \text{ with } \|x - E\| < \delta = \delta(\varepsilon, E). \tag{2}$$

Now we come to our main definition.

2.4. **Definition.** Let X be a convex set lying in a linear metric space Y. We say that X has the *locally convex approximation property* (the LCAP) iff there exist an F-norm $\| \ \|$ on Y, a sequence $\{X_n\}$ of LC-convex subsets of X, and a sequence of maps $r_n: X \to X_n$ such that for some summable sequence $\{a_n\}$ of positive numbers we have

$$\lim_{n\to\infty} \inf (a_n)^{-1}\|x - r_n(x)\| = 0 \quad \text{for every } x \in X. \tag{LC}$$

Of course every convex set in a locally convex space has the LCAP.

Our result in this section shows that the LCAP is in fact an extension of admissibility of Klee. Namely, we have the following theorem.

2.5. **Theorem.** *Any convex set with the LCAP is admissible. Conversely, any admissible compact convex set has the LCAP.*

Proof. Assume that X is a convex set with the LCAP. Let A be a compact subset of X, and let $\varepsilon > 0$. Using the notation of Definition 2.4, we take, for every $x \in A$, a neighbourhood $U(x)$ of x in A and $n(x) \in \mathbb{N}$ such that

$$(a_{n(x)})^{-1}\|y - r_{n(x)}(y)\| < (2a)^{-1}\varepsilon \quad \text{for every } y \in U(x), \tag{3}$$

where $a = \sum_{n=1}^{\infty} a_n$.

Since A is compact, there exist finite points $x_i \in A$, $i = 1, \dots, n$, such that $A = \bigcup_{i=1}^{n} U(x_i)$. Denote

$$\{n(1), \dots, n(k)\} = \{n(x_i): i = 1, \dots, n\}, \quad \text{where } n(1) < \cdots < n(k);$$
$$U_i = \bigcup\{U(x_j),\, n(x_j) = n(i)\}, \qquad i = 1, \dots, k;$$
$$f_i = r_{n(i)}: X \to X_{n(i)}, \qquad i = 1, \dots, k.$$

Then from (3) we get

$$\|x - f_i(x)\| < a_{n(i)}(2a)^{-1}\varepsilon \quad \text{for every } x \in U_i \text{ and } i = 1, \dots, k. \tag{4}$$

Observe that $\mathscr{U} = \{U_1, \dots, U_k\}$ is a finite open cover of A. Let $\{\lambda_1, \dots, \lambda_k\}$ be a partition of unity inscribed into $\mathscr{U}$.

Since $X_{n(i)}$ is LC-convex, there exists a $\delta_i > 0$ such that for every finite set $F \subset X_{n(i)}$ with $\operatorname{diam} F < \delta_i$ we have $\operatorname{diam} \operatorname{conv} F < a_{n(i)}(2a)^{-1}\varepsilon$.

Let $\mathscr{U}_i = \{U_{ij},\, j = 1, \dots, k(i)\}$ be a finite open cover of $f_i(A)$ such that

$$\operatorname{diam} U < 2^{-1}\delta_i \quad \text{for every } U \in \mathscr{U}_i.$$

Let $\{\lambda_{ij}: j = 1, \dots, k(i)\}$ be a partition of unity inscribed into $\mathscr{U}_i$. For every $j = 1, \dots, k(i)$, $i = 1, \dots, k$, select $x_{ij} \in U_{ij}$ and denote

$$D = \operatorname{conv}\{x_{ij},\, j = 1, \dots, k(i),\, i = 1, \dots, k\} \subset X.$$

We define a map $\varphi_i: f_i(A) \to D$ by the formula

$$\varphi_i(x) = \sum_{j=1}^{k(i)} \lambda_{ij}(x) x_{ij} \quad \text{for every } x \in f_i(A).$$

Then we have

$$\|\varphi_i(x) - x\| < a_{n(i)}(2a)^{-1}\varepsilon \quad \text{for every } x \in f_i(A). \tag{5}$$

Observe that D is a finite-dimensional compact convex set of X. We define a map $r: A \to D$ by the formula

$$r(x) = \sum_{i=1}^{k} \lambda_i(x)\varphi_i f_i(x) \quad \text{for every } x \in A.$$

Then from (4), (5) we get

$$\begin{aligned}
\|r(x) - x\| &= \left\| \sum_{i=1}^{k} \lambda_i(x)\varphi_i f_i(x) - x \right\| \\
&\le \left\| \sum_{i \in I(x)} \lambda_i(x)(\varphi_i f_i(x) - x) \right\| \quad (\text{where } I(x) = \{i : x \in U_i\}) \\
&\le \sum_{i \in I(x)} (\|\varphi_i f_i(x) - f_i(x)\| + \|f_i(x) - x\|) \\
&< 2 \sum_{i \in I(x)} a_{n(i)}(2a)^{-1}\varepsilon < 2 \sum_{n=1}^{\infty} a_n (2a)^{-1}\varepsilon = \varepsilon.
\end{aligned} \tag{6}$$

Consequently, X is admissible.

Conversely, assume that X is an admissible compact convex set. Then for every $n \in \mathbb{N}$ there exists a map f_n from X into a finite-dimensional convex subset X_n of X such that

$$\|f_n(x) - x\| < 2^{-n} \quad \text{for every } x \in X. \tag{7}$$

Since $\{a_n\} = \{n2^{-n}\}$ is a summable sequence, we infer that X has the LCAP. The theorem is proved.

2.6. *Remark.* In the next section we shall prove that any convex set with the LCAP is an AR. Therefore, from [BM] it follows that the compactness assumption in the converse part of Theorem 2.5 is *essential* unless all linear metric spaces are AR. In other words if there exists a non-AR linear metric space then there exists an admissible convex set which does not have the LCAP.

3. The AR-property for convex sets with the LCAP

In this section we prove that the LCAP does imply the AR-property for all convex sets (not necessarily separable) in linear metric spaces. Since every convex set in a locally convex space has the LCAP, our result is an extension of Dugundji theorem.

3.1. **Theorem.** *Any convex set with the LCAP is an AR.*

The proof of Theorem 3.1 is based on the following characterization of ANR-spaces (see [N1]).

Let X be a metric space. For an open cover $\mathscr{U}$ of X let $\mathscr{N}(\mathscr{U})$ denote the nerve of $\mathscr{U}$ equipped with *the Whitehead topology.* Let $\{\mathscr{U}_n\}$ be a sequence of open covers of X. We say that $\{\mathscr{U}_n\}$ is a *zero sequence* iff

$$\sup\{\operatorname{diam} U : U \in \mathscr{U}_n\} \to 0 \quad \text{as } n \to \infty. \tag{8}$$

We denote

$$\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n \quad \text{and} \quad \mathscr{K}(\mathscr{U}) = \bigcup_{n=1}^{\infty} \mathscr{N}(\mathscr{U}_n \cup \mathscr{U}_{n+1}). \tag{9}$$

For every $\sigma \in \mathscr{K}(\mathscr{U})$ we write

$$n(\sigma) = \sup\{n \in \mathbb{N}\colon \sigma \in \mathscr{N}(\mathscr{U}_n \cup \mathscr{U}_{n+1})\}. \tag{10}$$

We say that a map $f\colon \mathscr{U} \to X$ is a *selection* iff $f(U) \in U$ for every $U \in \mathscr{U}$.

The characterization of ANRs established by the author in [N1] is simplified to the following due to observations of J. Luukkainen and K. Sakai:

3.2. **Theorem** ([N1]; see also [NSk]). *A metric space X with no isolated points is an ANR if and only if there exists a zero sequence of open covers $\{\mathscr{U}_n\}$ of X such that for any selection $g\colon \mathscr{U} \to X$ there is a map $f\colon \mathscr{K}(\mathscr{U}) \to X$ such that for any sequence $\{\sigma_k\}$ of simplices of $\mathscr{K}(\mathscr{U})$ for which $n(\sigma_k) \to \infty$ we have* $\operatorname{diam}(f(\sigma_k) \cup g(\sigma_k^0)) \to 0$ *where σ^0 denotes the set of all vertices of σ.*

We need the following simple fact.

3.3. **Lemma.** *Let X be a convex set in a linear metric space, and let $r\colon X \to E$ be a map from X into an LC-convex subset E of X. Then for every $\varepsilon > 0$ there exists a family $\mathscr{U}$ of open subsets of X with the following properties:*

(i) $\operatorname{diam} U < \varepsilon$ *for every* $U \in \mathscr{U}$;

(ii) $\operatorname{diam} r(U) < 3^{-1}\delta$ *for every* $U \in \mathscr{U}$, *where* $\delta = \delta(\varepsilon, E) > 0$ (*see* 2.1);

(iii) $\|x - r(x)\| < 3^{-1}\varepsilon$ *if and only if* $x \in U$ *for some* $U \in \mathscr{U}$.

Proof. Let $\mathscr{V}$ be an open cover of E such that

$$\operatorname{diam} V < 3^{-1}\delta \quad \text{for every } V \in \mathscr{V}.$$

For every $V \in \mathscr{V}$ we denote

$$V_\varepsilon = \{x \in r^{-1}(V)\colon \|x - r(x)\| < 3^{-1}\varepsilon\}, \qquad \mathscr{U} = \{U\colon U = V_\varepsilon \text{ for some } V \in \mathscr{V}\}.$$

Since $\delta = \delta(\varepsilon, E) \le \varepsilon$, it is easy to see that $\mathscr{U}$ satisfies conditions (i)–(iii), and the lemma is proved.

Now we are already in a position to prove Theorem 3.1.

Assume that X is a convex set with the LCAP. Let $\{X_n\}$ be a sequence of LC-convex subsets of X, let $r_n\colon X \to X_n$ be a sequence of maps, and $\{a_n\}$ be a summable sequence of positive numbers satisfying condition (LC).

Applying Lemma 3.3 to $\varepsilon_n = a_n$ and $\delta_n = \delta(\varepsilon_n, X_n)$ we get a sequence $\{\mathscr{V}_n\}$ of families of open subsets of X with the following properties:

$$\operatorname{diam} V < \varepsilon_n = a_n \quad \text{for every } V \in \mathscr{V}_n; \tag{11}$$

$$\operatorname{diam} r_n(V) < 3^{-1}\delta_n \quad \text{for every } V \in \mathscr{V}_n; \tag{12}$$

$$\|x - r_n(x)\| < 3^{-1}\varepsilon_n \quad \text{if and only if} \quad x \in V \text{ for some } V \in \mathscr{V}_n. \tag{13}$$

For every $n \in \mathbb{N}$ we denote $\widehat{\mathscr{V}_n} = \bigcup_{i=n}^{\infty} \mathscr{V}_i$ and $\widehat{\mathscr{V}} = \widehat{\mathscr{V}_1}$.

We claim that $\{\widehat{\mathscr{V}_n}\}$ is a zero sequence of open covers of X. In fact, let $n \in \mathbb{N}$ and $x \in X$. By (LC) there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\|x - r_{n_k}(x)\| < 3^{-1}a_{n_k} = 3^{-1}\varepsilon_{n_k} \quad \text{for every } k \in \mathbb{N}.$$

Choose $k \in \mathbb{N}$ so that $n_k \geq n$. By (13) there exists $V \in \mathscr{V}_{n_k}$ such that $x \in V$. Whence, $V \in \widehat{\mathscr{V}_n}$, and from (11) we have

$$\sup\{\operatorname{diam} V \colon V \in \widehat{\mathscr{V}_n}\} \leq \sup\{a_i \colon i \geq n\} \to 0 \quad \text{as } n \to \infty.$$

Therefore $\{\widehat{\mathscr{V}_n}\}$ is a zero sequence and the claim is proved.

We shall check that the sequence $\{\widehat{\mathscr{V}_n}\}$ satisfies the conditions of Theorem 3.2. Observe that $\{\widehat{\mathscr{V}_n}\}$ is a decreasing sequence. Therefore

$$\widehat{\mathscr{V}} = \bigcup_{n=1}^{\infty} \widehat{\mathscr{V}_n} = \widehat{\mathscr{V}_1} \quad \text{and} \quad \mathscr{K}(\widehat{\mathscr{V}}) = \mathscr{N}(\widehat{\mathscr{V}_1}) \quad (\text{see } (10)).$$

And for each $\sigma \in \mathscr{K}(\widehat{\mathscr{V}})$, we have

$$n(\sigma) = \sup\{n \in \mathbb{N} \colon \sigma \in \mathscr{N}(\widehat{\mathscr{V}_n})\}.$$

Let $g \colon \widehat{\mathscr{V}} \to X$ be a selection. First we define a map $f \colon \widehat{\mathscr{V}} \to X$ as follows: For every $V \in \widehat{\mathscr{V}}$, we have $V \in \mathscr{V}_n$ for some $n \in \mathbb{N}$. We let $f(V) = r_n(g(V)) \in X_n$. Since $g(V) \in V$, from (13) we get

$$\|f(V) - g(V)\| = \|r_n(g(V)) - g(V)\| < 3^{-1}\varepsilon_n = 3^{-1}a_n. \tag{14}$$

Then we extend f by the convexity to the map, which is still denoted by f, $f \colon \mathscr{K}(\widehat{\mathscr{V}}) \to X$. We claim that f satisfies the condition of Theorem 3.2.

Observe that $f(\mathscr{K}(\widehat{\mathscr{V}})) \subset \bigcup_{n=1}^{\infty} \operatorname{conv}(X_n \cup X_{n+1})$.

First we compute $\operatorname{diam} f(\sigma)$. For every $\sigma = \langle V_1, \dots, V_k \rangle \in \mathscr{K}(\widehat{\mathscr{V}})$, take a finite sequence $\{m_i\}_{i=0}^{p} \subset \mathbb{N}$, with $m_0 = 0 \leq m_1 \leq \cdots \leq m_p = k$, such that

$$V_{m_i+1}, \dots, V_{m_{i+1}} \in \mathscr{V}_{n(\sigma)+i} \quad \text{for every } i = 0, 1, \dots, p-1, \tag{15}$$

and $f(V_j) = r_{n(\sigma)+i}(g(V_j)) \in X_{n(\sigma)+i}$ if $m_i < j \leq m_{i+1}$.

Observe that for every $x \in \operatorname{conv}\{f(V_j) \colon j = 1, \dots, k\}$, we have

$$x = \sum_{j=1}^{k} \lambda_j f(V_j) = \sum_{i=0}^{p-1} \sum_{j=m_i+1}^{m_{i+1}} \lambda_j f(V_j),$$

where $\lambda_j \geq 0$, $j = 1, \dots, k$, and $\sum_{j=1}^{k} \lambda_j = 1$.

Denote

$$\alpha_i = \sum_{j=m_i+1}^{m_{i+1}} \lambda_j \quad \text{for } i = 0, \dots, p-1,$$

and

$$\mu_{ij} = \begin{cases} (\alpha_i)^{-1}\lambda_j & \text{if } \alpha_i > 0, \\ 0 & \text{if } \alpha_i = 0. \end{cases}$$

Then we have

$$x = \sum_{i=0}^{p-1} \alpha_i \sum_{j=m_i+1}^{m_{i+1}} \mu_{ij} f(V_j),$$

where $\sum_{i=0}^{p-1} \alpha_i = 1$ and $\sum_{j=m_i+1}^{m_{i+1}} \mu_{ij} = 1$ for $i = 0, \dots, p-1$.

Note that $\bigcap_{j=1}^{k} V_j \neq \varnothing$. For every $i = 0, \dots, p-1$, select $b_i \in \bigcap_{j=m_i+1}^{m_{i+1}} V_j$ and denote $a_i = r_{n(\sigma)+i}(b_i) \in X_{n(\sigma)+i}$.

Take $a \in \bigcap_{j=1}^{k} V_j$. Observe that

$$
\begin{aligned}
x - a &= \sum_{i=0}^{p-1} \alpha_i \sum_{j=m_i+1}^{m_{i+1}} \mu_{ij}(f(V_j) - a_i) \\
&\quad + \sum_{i=0}^{p-1} \alpha_i \sum_{j=m_i+1}^{m_{i+1}} \mu_{ij}(a_i - a) \\
&= \sum_{i=0}^{p-1} \alpha_i \sum_{j=m_i+1}^{m_{i+1}} \mu_{ij}(f(V_j) - a_i) + \sum_{i=0}^{p-1} \alpha_i (a_i - a).
\end{aligned} \tag{16}
$$

Since a_i, $f(V_j) \in r_{n(\sigma)+i}(V_j)$ for $j = m_i + 1, \dots, m_{i+1}$, from (12) we get $\operatorname{diam}\{a_i, f(V_j), j = m_i+1, \dots, m_{i+1}\} < \delta_{n(\sigma)+i}$ for every $i = 0, 1, \dots, p-1$. Since $\delta_{n(\sigma)+i} = \delta(\varepsilon_{n(\sigma)+i}, X_{n(\sigma)+i})$ (see 2.1), we have $\operatorname{diam}\operatorname{conv}\{a_i, f(V_j) \colon j = m_i+1, \dots, m_{i+1}\} < \varepsilon_{n(\sigma)+i}$ for every $i = 0, 1, \dots, p-1$. From (11) and (13) we get

$$
\begin{aligned}
\|a - a_i\| &= \|a - r_{n(\sigma)+i}(b_i)\| \\
&\le \|a - b_i\| + \|b_i - r_{n(\sigma)+i}(b_i)\| \\
&< \varepsilon_{n(\sigma)+i} + 3^{-1}\varepsilon_{n(\sigma)+i} < 2\varepsilon_{n(\sigma)+i}.
\end{aligned}
$$

Hence from (16) we obtain

$$
\begin{aligned}
\operatorname{diam} f(\sigma) &= \operatorname{diam}\operatorname{conv}\{f(V_j) \colon j = 1, \dots, k\} \\
&\le 2\sup\{\|x - a\| \colon x \in \operatorname{conv}\{f(V_j) \colon j = 1, \dots, k\}\} \\
&\le 2\sum_{i=0}^{p-1}(\|a_i - a\| + \operatorname{diam}\operatorname{conv}\{a_i, f(V_j) \colon j = m_i + 1, \dots, m_{i+1}\}) \\
&\le 2\sum_{i=0}^{p-1}(2\varepsilon_{n(\sigma)+i} + \varepsilon_{n(\sigma)+i}) = 6\sum_{i=0}^{p-1}\varepsilon_{n(\sigma)+i} \\
&< 6\sum_{n=n(\sigma)}^{\infty}\varepsilon_n = 6\sum_{n=n(\sigma)}^{\infty} a_n.
\end{aligned}
$$

From (11) we get also

$$
\begin{aligned}
\operatorname{diam} g(\sigma^0) &\le 2\max\{\|g(V_j) - a\| \colon j = 1, \dots, k\} \\
&\le 2\max\{\operatorname{diam} V_j \colon j = 1, \dots, k\} \\
&< 2\sum_{n=n(\sigma)}^{\infty} a_n.
\end{aligned}
$$

Consequently, from (14) we get

$$
\begin{aligned}
\operatorname{diam}(f(\sigma) \cup g(\sigma^0)) &\le \operatorname{diam} f(\sigma) + 3^{-1}a_{n(\sigma)} + \operatorname{diam} g(\sigma^0) \\
&< 9\sum_{n=n(\sigma)}^{\infty} a_n.
\end{aligned}
$$

Since $\{a_n\}$ is a summable sequence, we have $\operatorname{diam}(f(\sigma) \cup g(\sigma^0)) \to 0$ as $n(\sigma) \to \infty$. Consequently, we have $X \in ANR$ by Theorem 3.2. Since X is contractible, it follows that $X \in AR$, and the proof of Theorem 3.1 is finished.

It is of interest to know that the LCAP implies the AR-property for all convex sets even in the nonseparable situation, meanwhile admissibility does not bring any information about the AR-property for noncompact convex sets. So the LCAP is really a "good" extension of admissibility.

From Theorem 3.1 and from Facts 2.2 and 2.3 we obtain

3.4. **Corollary.** *Any σ-LC-convex set is an AR.*

Here we say that a convex set X is σ-LC-convex iff X is a union of an increasing sequence of LC-convex subsets of X.

Proof. Assume that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}$ is an increasing sequence of LC-convex subsets of X. By 2.2 we may assume that X_n is closed in X for every $n \in \mathbb{N}$. By 2.3 for every $n \in \mathbb{N}$ there exists a Dugundji retraction $r_n: X \to X_n$. Since $\{X_n\}$ is an increasing sequence, for every $x \in X$ there exists an $n(x) \in \mathbb{N}$ such that $r_n(x) = x$ for every $n \geq n(x)$. Consequently, X has the LCAP, therefore is an AR by Theorem 3.1. The corollary is proved.

Obviously every separable convex set contains a dense σ-LC convex subset. Therefore, from Corollary 3.4 it follows that an affirmative answer to the following question would imply the AR-property for all separable convex sets.

3.5. **Question.** Let $\overline{X}$ denote the closure of a convex set X lying in a linear metric space. Assume that X has the LCAP. Has $\overline{X}$ the LCAP?

Observe that Theorem 3.1 reduces Problem 1.1 to

3.6. **Question.** Has every convex set the LCAP? What about separable convex sets? Compact convex sets?

In [N2] we introduced the notion of *the finite-dimensional approximation property* (the FDAP) for convex sets in linear metric spaces and proved that if X is a convex set with the FDAP, then every convex subset of X is an AR. We do not know whether the LCAP has this stronger property.

3.7. **Question.** Assume that X is a convex set with the LCAP. Is every convex subset of X an AR?

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