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YOUNGER MATES AND THE JACOBIAN CONJECTURE

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ABSTRACT. Let F, $G \in \mathbb{C}[x, y]$. If the Jacobian determinant of F and G is 1, then G is said to be a *Jacobian mate* of F. If, in addition, G has degree less than that of F, then G is said to be a *younger mate* of F. In this paper, a necessary and sufficient condition is given for a polynomial to have a younger mate. This also gives rise to a formula for the younger mate if it exists. Furthermore, a conjecture concerning the existence of a younger mate is shown to be equivalent to the Jacobian conjecture.

Throughout this paper, F and G will be polynomials in $\mathbb{C}[x, y]$ where $\mathbb{C}$ denotes the field of complex numbers. We say that F and G satisfy the *Jacobian hypothesis* if their Jacobian determinant is one, i.e., $F_x G_y - F_y G_x = 1$. In this case, we also say that G is a *Jacobian mate* of F. Furthermore, if the x-degree (resp. y-degree, total degree) of G is less than that of F, then G is said to be a *younger mate* of F relative to the x-degree (resp. y-degree, total degree). For instance, $x + y$ has younger mates y and $-x$ relative to the x-degree and the y-degree, respectively, but has no younger mate relative to the total degree.

This paper was motivated by the *Jacobian conjecture* which asserts that if F has a Jacobian mate G, then (F, G) is an automorphism pair. In Section 1, it is shown that a younger mate is unique (up to an additive constant) and universal, i.e., if a Jacobian mate G of F exists, then any other mate of F can be expressed as G plus a polynomial in F. In Section 2, the problem of existence of a younger mate of F is reduced to the case where F is monic in both variables. In Section 3, a necessary and sufficient condition for the existence of a younger mate and a formula for a younger mate provided one exists are given. Finally, in Section 4, a conjecture concerning the existence of younger mates is formulated and shown to be equivalent to the Jacobian conjecture.

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1. Properties of Jacobian mates

Following [12, Theorem 33, p. 472], we shall use $[\![F, G]\!]$ to denote the Jacobian determinant. Note that $[\![\ ,\ ]\!]$, as a function from $\mathbb{C}[x, y] \times \mathbb{C}[x, y]$ to $\mathbb{C}[x, y]$, is bilinear.

Theorem 1. *Suppose* $[\![F, G]\!] = 1$ *and* $[\![F, K]\!] = 0$ *for* F, G, K *in* $\mathbb{C}[x, y]$. *Then* $K \in \mathbb{C}[F]$.

Proof. Since $[\![F, G]\!] = 1$, F and G are algebraically independent over $\mathbb{C}$. Moreover, any $\mathbb{C}$-derivation on $\mathbb{C}[F, G]$ extends uniquely to a $\mathbb{C}$-derivation on $\mathbb{C}[x, y]$ by [13, Theorem 1, (1), (5), pp. 240–241]. In particular, $\frac{\partial}{\partial G}: \mathbb{C}[F, G] \to \mathbb{C}[F, G]$ extends uniquely to $D: \mathbb{C}[x, y] \to \mathbb{C}[x, y]$. Indeed, the derivation D is given by $D(H) = [\![F, H]\!]$ for $H \in \mathbb{C}[x, y]$ by [13, proof of (5) $\Longrightarrow$ (2) of Theorem 1, p. 241]. Then by [2, (1.5) Corollary, p. 74], $\mathbb{C}[F] = \ker(D)$. Since $D(K) = 0$,

$$K \in \ker(D) = \mathbb{C}[F]. \quad \square$$

Corollary 2 (Universality of Jacobian Mate). *Suppose* $[\![F, G]\!] = 1$ *and* $[\![F, H]\!] = 1$ *for* F, G, H *in* $C[x, y]$. *Then* $G - H \in \mathbb{C}[F]$.

Proof. Apply Theorem 1 with $K = G - H$. $\square$

Theorem 3 (Uniqueness of Younger Mate). *If a younger mate relative to the* x*-degree* (*resp.* y*-degree, total degree*) *exists, then it is unique up to an additive constant.*

Proof. Suppose both G and H are younger mates of F relative to the x-degree. Then

$$[\![F, G - H]\!] = [\![F, G]\!] - [\![F, H]\!] = 1 - 1 = 0.$$

By Corollary 2, $G - H = \phi(F) \in \mathbb{C}[F]$. Since $\deg_x(G - H) < \deg_x F$, $\phi(F)$ is a constant. The proofs for y-degree and total degree are similar. $\square$

2. Reduction

In this section we shall reduce the problem of determining the existence of a younger mate of F to the case where F is monic in both x and y. Recall that the Newton polygon for $F(x, y)$ is the convex hull of the origin together with the support of F.

Lemma 4. *If* $\deg F \geqslant 2$, $\deg G \geqslant 2$ *and* $[\![F, G]\!] = 1$, *then the Newton polygons of* F *and* G *are similar.* $\square$

The first proof of this lemma was given by Oka [10, Lemma (6.1), p. 430]. The second proof was given by Applegate and Onishi [1, 14. Lemma, p. 217]. Because these proofs were incomplete, Nowicki and Nakai [8, Lemma B, p. 305] offered the third proof. However, as pointed out by L. Andrew Campbell in his review [Mathematical Reviews 89h:13007] of the article, this new proof still contained a gap. Moreover, M. Nagata also indicated an error of the new proof to the authors, so Nowicki and Nakai corrected this in [9].

However, we shall need the following stronger version which can be derived from Lemma 4 and its various proofs.

Lemma 5. *If* $\deg_x F \geqslant 1$, $\deg_y F \geqslant 1$, $\deg_x G \geqslant 1$, $\deg_y G \geqslant 1$, *and* $[\![F, G]\!] = 1$, *then the Newton polygons of* F *and* G *are similar with magnification factor equal to*

$$\frac{\deg_x F}{\deg_x G} = \frac{\deg_y F}{\deg_y G} = \frac{\deg F}{\deg G}. \quad \square$$

We have indicated in the beginning that a younger mate of F relative to the x-degree (resp. y-degree) may not be a younger mate relative to the total degree. However, the next proposition shows that all three concepts coincide for a "nondegenerate" F.

Proposition 6. *If* $\deg_x F \geqslant 2$, $\deg_y F \geqslant 2$ *and* $[\![F, G]\!] = 1$, *then the following conditions are equivalent.*

(1) $\deg_x G < \deg_x F$.
(2) $\deg_y G < \deg_y F$.
(3) $\deg G < \deg F$.

Proof. We first claim that $\deg_x G \geqslant 1$. Otherwise $G_x = 0$, and the Jacobian hypothesis implies that $F_x \in \mathbb{C} \setminus \{0\}$. Hence $F = ax + p(y)$ for some nonzero constant a and some polynomial $p(y) \in \mathbb{C}[y]$. This contradicts the assumption that $\deg_x F \geqslant 2$. Likewise $\deg_y G \geqslant 1$. Now the result follows from Lemma 5. $\square$

$F(x, y)$ is said to be *monic* in x if either it is a polynomial in y alone or it is of the form

$$F(x, y) = x^n + p_{n-1}(y)\, x^{n-1} + \cdots + p_1(y)\, x + p_0(y) \tag{3.1}$$

where every $p_i(y) \in \mathbb{C}[y]$ and $n \geqslant 1$. For any polynomial $F(x, y)$, there is an invertible linear substitution

$$\begin{cases} x = \alpha\bar{x} + \beta\bar{y}, \\ y = \gamma\bar{x} + \delta\bar{y} \end{cases}$$

(so that α, β, γ, $\delta \in \mathbb{C}$ and $\alpha\delta - \beta\gamma \neq 0$) such that

$$\bar{F}(\bar{x}, \bar{y}) = F(\alpha\bar{x} + \beta\bar{y}, \gamma\bar{x} + \delta\bar{y})$$

is monic in $\bar{x}$ and $\bar{y}$ and $\deg F = \deg_{\bar{x}} \bar{F} = \deg_{\bar{y}} \bar{F}$ [5, (2.7), p. 5]. Note that the above substitution does not change the total degree nor the existence of a Jacobian mate of any polynomial. In the following we shall reserve $\bar{F}$ for the result of this transformation on the fixed polynomial F.

Proposition 7. *Suppose* $\deg F \geqslant 2$. *Then the following conditions are equivalent.*

(1) F *has a younger mate relative to the total degree.*
(2) $\bar{F}$ *has a younger mate relative to the* $\bar{x}$*-degree.*

Proof. (1) $\Longrightarrow$ (2). Note that this implication does not need the hypothesis that $\deg F \geqslant 2$. Suppose G is a younger mate of F relative to the total degree. Let $\bar{G}(\bar{x}, \bar{y}) = G(\alpha\bar{x} + \beta\bar{y}, \gamma\bar{x} + \delta\bar{y})$. Then, by the chain rule for Jacobians, $\frac{1}{\alpha\delta - \beta\gamma}\bar{G}$ is a Jacobian mate for $\bar{F}$. Moreover, $\deg_{\bar{x}} \bar{G} \leqslant \deg \bar{G} = \deg G < \deg F = \deg_{\bar{x}} \bar{F}$.

(2) $\Longrightarrow$ (1). Suppose H is a younger mate of $\bar{F}$ relative to the $\bar{x}$-degree. Let $G(x, y) = H(\bar{\alpha}x + \bar{\beta}y, \bar{\gamma}x + \bar{\delta}y)$ where

$$\begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1}.$$

Then, by the chain rule for Jacobians, $(\alpha\delta - \beta\gamma)G$ is a Jacobian mate of F. Since $\deg_{\bar{x}} \bar{F} = \deg_{\bar{y}} \bar{F} = \deg F \geqslant 2$, we may apply Proposition 6 to $\bar{F}$ and H to obtain $\deg H < \deg \bar{F}$. Thus $\deg G < \deg F$. □

Combining Propositions 6 and 7, we see that if $\deg_x F \geqslant 2$ and $\deg_y F \geqslant 2$, then F has a younger mate relative to any degree if and only if $\bar{F}$ has a younger mate relative to the $\bar{x}$-degree. In the next section we shall derive a necessary and sufficient condition for the latter.

3. Existence of younger mates

Lemma 8. *Suppose that ν is a ring homomorphism from R to S where R is a commutative ring and S is an algebraically closed field. Let H and K be in $R[x]$. Then $\nu(\mathrm{Res}_x(H, K)) = 0$ if and only if either $\deg_x \tilde{\nu}(H) < \deg_x H$ and $\deg_x \tilde{\nu}(K) < \deg_x K$, or $\tilde{\nu}(H)$ and $\tilde{\nu}(K)$ have a common linear factor in $S[x]$. Here $\tilde{\nu}: R[x] \to S[x]$ is the natural extension of ν.*

Proof. The nondegenerate case is proved in [11, Chapter 5, Section 8, p. 104, lines 2 to 5]. □

Proposition 9. *Assume that F is monic in x with x-degree $\geqslant 1$. If F has a Jacobian mate, then the resultant of F_x and F_y with respect to x must be a nonzero constant.*

Proof. To show that $r(y) = \mathrm{Res}_x(F_x, F_y)$ is a nonzero constant, it suffices to show that $r(y)$ has no roots. Suppose $r(y)$ has a root y_0 in $\mathbb{C}$. Applying Lemma 8 with $R = \mathbb{C}[y]$, $S = \mathbb{C}$ and ν being the evaluation map at y_0, we see that $F_x(x, y_0)$ and $F_y(x, y_0)$ have a common linear factor in $\mathbb{C}[x]$ since

$$\deg_x F_x(x, y_0) = \deg_x F_x(x, y) = n - 1.$$

Let the common factor be $x - x_0$. Then $F_x(x_0, y_0) = F_y(x_0, y_0) = 0$. However, setting $x = x_0$ and $y = y_0$ in the Jacobian hypothesis leads to $0 = 1$. □

Suppose $F(x, y)$ is of the form (3.1) with $n \geqslant 2$ and $\deg_y F \geqslant 1$. Let k be the largest i such that $p_i(y)$ is a nonconstant. Since $\deg_y F \geqslant 1$, k exists. Therefore

$$\begin{aligned} F_x(x, y) &= n x^{n-1} + (n-1)p_{n-1}(y) x^{n-2} + \cdots + p_1(y), \\ F_y(x, y) &= p_k'(y) x^k + p_{k-1}'(y) x^{k-1} + \cdots + p_0'(y). \end{aligned}$$

Since $n \neq 0$ and $p_k'(y) \neq 0$, $r(y) = \mathrm{Res}_x(F_x, F_y) \in \mathbb{C}[y]$ is the determinant of

the following $(k+n-1)\times(k+n-1)$ Sylvester matrix $\mathbf{S}$:

$$\left[\begin{array}{ccccccccc}
n & (n-1)p_{n-1} & \cdots & \cdots & p_1 & & & & \\
 & n & (n-1)p_{n-1} & \cdots & \cdots & p_1 & & & \\
 & & \ddots & \ddots & & & \ddots & & \\
 & & & \ddots & \ddots & & & \ddots & \\
 & & & & n & (n-1)p_{n-1} & \cdots & \cdots & p_1 \\
p'_k & p'_{k-1} & \cdots & \cdots & \cdots & p'_0 & & & \\
 & p'_k & p'_{k-1} & \cdots & \cdots & \cdots & p'_0 & & \\
 & & \ddots & \ddots & & & & \ddots & \\
 & & & p'_k & p'_{k-1} & \cdots & \cdots & \cdots & p'_0
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}n\\n\\n\\n\\n\end{array}}\right\} k \text{ rows} \\
\left.\vphantom{\begin{array}{c}n\\n\\n\\n\end{array}}\right\} n-1 \text{ rows}
\end{array}$$

Let $\mathbf{S}^{\mathrm{adj}}$ denote the adjoint of $\mathbf{S}$. If the bottom row of $\mathbf{S}^{\mathrm{adj}}$ has entries $a_1(y)$, $a_2(y), \ldots, a_{k+n-1}(y)$, let

$$\tilde{A}(x, y) = a_1(y)\,x^{k-1} + a_2(y)\,x^{k-2} + \cdots + a_k(y),$$
$$\tilde{B}(x, y) = -\Big(a_{k+1}(y)\,x^{n-2} + a_{k+2}(y)\,x^{n-3} + \cdots + a_{k+n-1}(y)\Big).$$

Theorem 10. *Suppose $F(x, y)$ is monic in x with $\deg_x F \geqslant 2$ and $\deg_y F \geqslant 1$. Let $c = \mathrm{Res}_x(F_x, F_y)$. Then the following two conditions are equivalent.*

(1) *F has a younger mate relative to the x-degree.*
(2) *$c \in \mathbb{C} \setminus \{0\}$ and $\tilde{A}_x = \tilde{B}_y$.*

If these two conditions are satisfied, then the younger mate G of F relative to the x-degree is given by

$$G = \frac{1}{c}\left(\int \tilde{B}\,dx + \int \tilde{A}\,dy - \iint \tilde{A}_x\,dy\,dx\right).$$

Proof. Multiplying both sides of the matrix equation

$$\mathbf{S} \times \begin{bmatrix} x^{k+n-2} \\ \vdots \\ x^{n-1} \\ x^{n-2} \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x^{k-1}F_x \\ \vdots \\ F_x \\ x^{n-2}F_y \\ \vdots \\ F_y \end{bmatrix}$$

on the left by the adjoint of $\mathbf{S}$, we have

$$\begin{bmatrix} c & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & c \end{bmatrix} \times \begin{bmatrix} x^{k+n-2} \\ \vdots \\ x^{n-1} \\ x^{n-2} \\ \vdots \\ 1 \end{bmatrix} = \mathbf{S}^{\mathrm{adj}} \times \begin{bmatrix} x^{k-1}F_x \\ \vdots \\ F_x \\ x^{n-2}F_y \\ \vdots \\ F_y \end{bmatrix}$$

where $c = \det \mathbf{S}$. By comparing the bottom rows of both sides, we have

$$\begin{aligned} c &= a_1(y)\,x^{k-1}F_x + a_2(y)\,x^{k-2}F_x + \cdots + a_k(y)F_x \\ &\quad + a_{k+1}(y)\,x^{n-2}F_y + a_{k+2}(y)\,x^{n-3}F_y + \cdots + a_{k+n-1}(y)F_y \\ &= \tilde{A}(x\,,\,y)\,F_x - \tilde{B}(x\,,\,y)\,F_y. \end{aligned}$$

(1) $\Longrightarrow$ (2). If F has a younger mate relative to the x-degree, then $c = \mathrm{Res}_x(F_x\,,\,F_y)$ must be a nonzero constant by Proposition 9. The last equation becomes

$$1 = A\,F_x - B\,F_y \tag{4.1}$$

with $B = \frac{1}{c}\tilde{B}$ and $A = \frac{1}{c}\tilde{A}$. Now let G be a younger mate of F relative to the x-degree. Consider the matrix

$$V = \begin{bmatrix} B & B & A \\ F_x & F_x & F_y \\ G_x & G_x & G_y \end{bmatrix}.$$

Since the first two columns are identical, $\det V = 0$. On the other hand, expanding the determinant of V according to the first column and using (4.1) and $[\![F\,,\,G]\!] = 1$, we have $\det V = B[\![F\,,\,G]\!] - F_x(G_yB - G_xA) + G_x(F_yB - F_xA) = B + F_x(G_xA - G_yB) - G_x$. Hence $G_x - B = F_x(G_xA - G_yB)$. Likewise, $G_y - A = F_y(G_xA - G_yB)$. So we have

$$\begin{cases} G_x - B = F_x\,(G_xA - G_yB)\,, \\ G_y - A = F_y\,(G_xA - G_yB). \end{cases} \tag{4.2}$$

We claim that $G_x - B = 0$. Otherwise (4.2) implies that

$$\deg_x(G_x - B) = \deg_x F_x + \deg_x(G_xA - G_yB) \geqslant \deg_x F_x.$$

On the other hand, $\deg_x G < \deg_x F$. This, together with $\deg_x B \leqslant n-2 < n-1 = \deg_x F_x$, shows that $\deg_x(G_x - B) < \deg_x F_x$, a contradiction. Therefore $G_x - B = 0$. By the first equation of (4.2), we have $G_xA - G_yB = 0$ since $F_x \neq 0$. Hence $G_y - A = 0$ follows from the second equation of (4.2). Thus $G_{yx} - A_x = 0$ and $G_{xy} - B_y = 0$.

(2) $\Longrightarrow$ (1). Assume that c is a nonzero constant and $\tilde{A}_x = \tilde{B}_y$. Let $B = \frac{1}{c}\tilde{B}$ and $A = \frac{1}{c}\tilde{A}$. Then $A_x = B_y$ and so there exists a polynomial $G(x\,,\,y) \in \mathbb{C}[x\,,\,y]$ such that $G_x = B$ and $G_y = A$. Then the x-degree of G is at most $n-1$, and (4.1) becomes $1 = [\![F\,,\,G]\!]$. $\square$

Remark. If $\deg_x F = 1$, F does not have a younger mate relative to the x-degree.

4. The younger mate conjecture

In this section we formulate the following conjecture and show that it is equivalent to the Jacobian conjecture.

The younger mate conjecture. *Suppose that F has a Jacobian mate and $\deg F \geqslant 2$. Then there exists G_1 such that $[\![F, G_1]\!] = 1$ and $\deg G_1 < \deg F$. In other words, if F has a Jacobian mate, then it has a younger mate relative to the total degree.* □

We say that (F, G) is an *automorphism pair* if the $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[x, y]$ to $\mathbb{C}[x, y]$ determined by $x \mapsto F$ and $y \mapsto G$ is an automorphism.

Lemma 11. *Suppose $[\![F, G]\!] = 1$ and $[\![F, H]\!] = 1$. Then (F, G) is an automorphism pair if and only if (F, H) is also an automorphism pair.*

Proof. By Corollary 2, $(F, G) = \big(F, H+\phi(F)\big) = \big(x, y+\phi(x)\big)\circ(F, H)$. Now the result follows from the fact that $(x, y+\phi(x))$ is an automorphism pair. □

Lemma 12. *If (F, G) is an automorphism pair with $\deg F \leqslant \deg G$, then $\deg F$ divides $\deg G$.*

Proof. For an elementary proof, as well as a historical account, see [7]. □

Lemma 13 (Dixmier [4, 2.7. Lemme, p. 215]). *If $[\![f, g]\!] = 0$ where $f, g \in \mathbb{C}[x, y]$ are nonzero and homogeneous, then $\frac{f^{\deg g}}{g^{\deg f}} \in \mathbb{C} \setminus \{0\}$.* □

Corollary 14. *Suppose (F, G) is an automorphism pair with $\deg F \leqslant \deg G$. Then there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\deg G_1 < \deg G$ and $G_1 = G - \lambda F^{\deg G/\deg F}$.*

Proof. The case in which $\deg F = \deg G = 1$ is easy to see, so we only consider the case where $\deg G \geqslant 2$. Let F^+ and G^+ be the highest homogeneous part of F and G respectively. Since (F, G) is an automorphism pair, $[\![F, G]\!] \in \mathbb{C} \setminus \{0\}$ by the chain rule for Jacobians. Using the biadditivity of the Jacobian and counting degrees, we obtain $[\![F^+, G^+]\!] = 0$. By Lemma 13, $(F^+)^{\deg G}/(G^+)^{\deg F} \in \mathbb{C} \setminus \{0\}$. Since $\deg F$ divides $\deg G$ by Lemma 12, it follows that $(F^+)^{\deg G/\deg F}/G^+ \in \mathbb{C} \setminus \{0\}$ and so there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $G^+ = \lambda(F^+)^{\deg G/\deg F}$. Hence the result follows. □

Lemma 15. *Suppose $[\![F, G]\!] = 1$ with $\deg G = 1$. Then (F, G) is an automorphism pair.*

Proof. After a suitable change of coordinates, we may assume that $G = y$. Then $[\![F, G]\!] = 1$ implies that $F_x = 1$, i.e., $F = x + p(y)$ for some $p(y) \in \mathbb{C}[y]$. □

Theorem 16. *The younger mate conjecture is equivalent to the Jacobian conjecture.*

Proof. $\Longrightarrow$: Suppose the younger mate conjecture is true, and suppose $[\![F, G]\!] = 1$. Then there exists G_1 such that $[\![F, G_1]\!] = 1$ and $\deg G_1 < \deg F$. Hence $[\![G_1, -F]\!] = 1$ and, by the younger mate conjecture, there exists G_2 such that

$[\![G_1, G_2]\!] = 1$ and $\deg G_2 < \deg G_1$ provided that $\deg G_1 \geqslant 2$. Repeating this process, we obtain a decreasing sequence

$$\deg F > \deg G_1 > \deg G_2 > \cdots$$

such that

$$[\![G_i, G_{i+1}]\!] = 1$$

for $i \geqslant 0$ where $F = G_0$. This sequence must stop, say, at G_s. Then $\deg G_s \leqslant 1$ and hence $\deg G_s = 1$ by $[\![G_{s-1}, G_s]\!] = 1$. By Lemma 11, (F, G) is an automorphism pair if and only if (F, G_1) is. Similarly, as $[\![G_{i+1}, G_{i+2}]\!] = [\![G_{i+1}, -G_i]\!] = 1$, (G_{i+1}, G_{i+2}) is an automorphism pair if and only if $(G_{i+1}, -G_i)$ is. The latter is equivalent to saying that (G_i, G_{i+1}) is an automorphism pair. So (F, G) is an automorphism pair if and only if (G_{s-1}, G_s) is. But (G_{s-1}, G_s) is an automorphism pair by Lemma 15.

$\Longleftarrow$: Suppose the Jacobian conjecture is true, and suppose that $[\![F, G]\!] = 1$ with $\deg F \geqslant 2$. We may assume that $\deg F \leqslant \deg G$ for otherwise we are done. Then the Jacobian conjecture implies that (F, G) is an automorphism pair. By Corollary 14, there exists G_1 such that $\deg G_1 < \deg G$ and $[\![F, G_1]\!] = 1$. If $\deg G_1 \geqslant \deg F$, then using Corollary 14 again, we obtain G_2 such that $\deg G_2 < \deg G$ and $[\![F, G_2]\!] = 1$. Repeat this process to create a decreasing sequence

$$\deg G > \deg G_1 > \deg G_2 > \cdots$$

such that

$$[\![F, G_i]\!] = 1$$

for all $i \geqslant 1$. The sequence must stop, say, at G_s. Then $[\![F, G_s]\!] = 1$ and $\deg G_s < \deg F$. $\square$

Remark. If the conclusion $\deg G_1 < \deg F$ in the younger mate conjecture is replaced by $\deg G_1 < \deg G$, then it becomes false (although it is easy to see that the result is stronger than before). For example, if $F = y + (x + y^2)^2$ and $G = x + y^2$, then such G_1 does not exist: As $\deg G_1 < \deg G = 2$, we may write $G_1 = \alpha x + \beta y + s$ where $\alpha, \beta, s \in \mathbb{C}$; then $[\![F, G_1]\!] = -\alpha + 2(x+y^2)(\beta - 2\alpha y) \neq 1$.

Addendum. Yosef Stein has also announced some conditions for existence of a Jacobian mate in his talk entitled "The Jacobian problem in two variables as a system of ordinary differential equations" in the International Conference on Polynomial Automorphisms and Related Topics held in CIRM (Centre International de Rencontres Mathématiques), Luminy, Marseille, France in October 1992.

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