PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 10, October 1995

STATISTICS FOR SPECIAL q, t-KOSTKA POLYNOMIALS

SUSANNA FISHEL

(Communicated by Jeffry N. Kahn)

ABSTRACT. Kirillov and Reshetikhin introduced rigged configurations as a new way to calculate the entries $K_{\lambda\mu}(t)$ of the Kostka matrix. Macdonald defined the two-parameter Kostka matrix whose entries $K_{\lambda\mu}(q, t)$ generalize $K_{\lambda\mu}(t)$. We use rigged configurations and a formula of Stembridge to provide a combinatorial interpretation of $K_{\lambda\mu}(q, t)$ in the case where μ is a partition with no more than two columns. In particular, we show that in this case, $K_{\lambda\mu}(q, t)$ has nonnegative coefficients.

1. INTRODUCTION

In [Mac2], Macdonald defined a basis $P_\lambda(q, t)$ of the ring of symmetric functions. Hall–Littlewood symmetric functions, Jack polynomials, Schur functions, and zonal polynomials are all either limiting or special cases of the $P_\lambda(q, t)$'s. He also defined a transition matrix, whose entries are denoted $K_{\lambda\mu}(q, t)$, between a renormalized version of the $P_\lambda(q, t)$'s and another basis S_λ of the ring of symmetric functions. $(K_{\lambda\mu}(q, t))_{\lambda, \mu\vdash n}$ generalizes the Kostka matrix $(K_{\lambda\mu})_{\lambda, \mu\vdash n}$.

Macdonald conjectured that the entries in the two-parameter Kostka matrix are polynomials in q and t with nonnegative integer coefficients. All that is known a priori is that the entries are rational functions of q and t. Garsia and Haiman have constructed, for each partition μ of n, a finite-dimensional bigraded S_n-module whose irreducible multiplicities they conjecture to be rescaled versions of the entries $K_{\lambda\mu}(q, t)$. In [GH], they give several constructions of S_n-modules conjectured to have this property, together with an announcement of the special cases for which they can prove their conjecture. The main special cases correspond to the entries $K_{\lambda\mu}(q, t)$ in which μ is either a hook, or has at most two rows or two columns. Their results do not provide any explicit combinatorial interpretation of the entries.

In the paper [Ste], Stembridge gave a direct proof of the hook case of Macdonald's conjecture, and gave a formula for the polynomial in the two-column

Received by the editors October 14, 1993 and, in revised form, March 7, 1994.
1991 *Mathematics Subject Classification*. Primary 05E05.
Key words and phrases. Two-parameter Kostka matrix, rigged configurations.

case, which proves the entries are polynmials. His formula is

$$K_{\lambda 2^r 1^{n-2r}}(q\,,\,t) = \sum_{s=0}^{r} q^{r-s}(t^{n-r}q\,;\,t^{-1})_s \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t) \tag{1}$$

where $K_{\lambda 2^s 1^{n-2s}}(t)$ is the Kostka (charge) polynomial [Mac1] and $|\lambda| = n$. We use (1) to show that there are statistics c_r and cut_r defined on the set $\mathcal{M}_0^0$ of Kirillov and Reshetikhin rigged configurations which correspond to standard Young tableaux of shape λ such that

Theorem 1.1.

$$K_{\lambda 2^r 1^{n-2r}}(q\,,\,t) = \sum_{(\alpha(0)\,,\,L)\in\mathcal{M}_0^0} q^{cut_r(\alpha(0)\,,\,L)} t^{c_r(\alpha(0)\,,\,L)}. \tag{2}$$

Theorem 1.1 proves that $K_{\lambda 2^r 1^{n-2r}}(q\,,\,t)$ has nonnegative coefficients.

This paper is divided into five sections. In Section 2, we explain the necessary Kirillov and Reshetikhin material and introduce notation. In Section 3 we rewrite $K_{\lambda 2^r 1^{n-2r}}(q\,,\,t)$ as a sum of "difference" polynomials. In Section 4 we show the difference polynomials are nonnegative by showing they are generating functions for sets $\mathcal{M}_m^d$ of rigged configurations. In Section 5 we finish proving Theorem 1.1.

2. Kirillov and Reshetikhin's rigged configurations

Kirillov and Reshetikhin [KR1, KR2] introduced rigged configrations as a new way to calculate $K_{\lambda\mu}(t)$ for any pair of partitions λ and μ of n. Fix n, a positive integer, and λ, a partition of n. Let $\alpha = (\alpha^1\,,\,\alpha^2\,,\,\dots\,,\,\alpha^x)$ be a sequence of partitions such that $|\alpha^i| = \lambda_{i+1} + \lambda_{i+2} + \cdots$. For any such sequence, if μ is any partition of n, let $\alpha(\mu)$ be the sequence of partitions $(\mu' = \alpha^0\,,\,\alpha^1\,,\,\dots\,,\,\alpha^x)$, and if m is a nonnegative integer, let $\alpha(m) = ((n-m\,,\,m)\,,\,\alpha^1\,,\,\alpha^2\,,\,\dots\,,\,\alpha^x)$. $\alpha(\mu)$ is called a configuration.

Define, for $k \geq 1$,

$$P_l^k(\alpha(\mu)) = \sum_{i=1}^{l}(\alpha_i^{k-1} - 2\alpha_i^k + \alpha_i^{k+1}). \tag{3}$$

A rigged configuration is a pair $(\alpha(\mu)\,,\,L)$, where L labels the columns of the partitions in α. In particular,

(1) $0 \leq L_i^k \leq P_l^k(\alpha(\mu))$ for $k \geq 1$ and $1 \leq i \leq \alpha_1^k$ where the length of column i of α^k is l,

(2) If column i of α^k has the same length as column $i+1$, then $L_i^k \leq L_{i+1}^k$.

Not all sequences $\alpha(\mu)$ will have labels. Kirillov and Reshetikhin call a sequence $\alpha(\mu)$ a μ-admissible λ configuration when there is at least one labelling function L, that is, when $P_i^k(\alpha(\mu)) \geq 0$. They have defined a bijection between μ-admissible λ rigged configurations and column strict tableaux of shape λ and content μ.

Further, Kirillov and Reshetikhin define the charge of a rigged configuration $c((\alpha(\mu), L))$. Let

$$c(\alpha(\mu)) = n(\mu) - \sum_{i\geq 1} \mu_i' \alpha_i^1 + \sum_{k, i\geq 1} \alpha_i^k(\alpha_i^k - \alpha_i^{k+1}),$$

where $n(\mu) = \sum(i-1)\mu_i$. Then

$$c((\alpha(\mu), L)) = c(\alpha(\mu)) + \sum_{i, k\geq 1} L_i^k.$$

The Kirillov and Reshetikhin theorem is now

$$K_{\lambda\mu}(t) = \sum_{(\alpha(\mu), L)} t^{c((\alpha(\mu), L))}$$

where the sum is over all μ-admissible λ rigged configurations.

In this paper, $\mu = (2^m 1^{n-2m})$, so that $\alpha(\mu) = \alpha(m)$. We need several properties which are peculiar to this case.

(1) Let

$$c'(\alpha) = \sum_{k, i\geq 1} \alpha_i^k(\alpha_i^k - \alpha_i^{k+1}). \tag{4}$$

Then

$$c((\alpha(m))) = \binom{n-m}{2} + \binom{m}{2} - (n-m)\alpha_1^1 - m\alpha_2^1 + c'(\alpha). \tag{5}$$

(2) By the definition of $P_i^k(\alpha(m))$

$$P_i^k(\alpha(m+1)) = \begin{cases} P_i^k(\alpha(m)) - 1 & \text{if } i = k = 1, \\ P_i^k(\alpha(m)) & \text{otherwise.} \end{cases} \tag{6}$$

Finally, let $\mathscr{M}_m^0 = \{(\alpha(m), L) | \alpha(m)$ is a $(n-m, m)$-admissable λ configuration and L is a label$\}$. Call $\mathscr{M}_0^0$ the set of standard rigged configurations.

3. Difference polynomials

In this section, we rewrite Stembridge's formula (1), changing it to (8). Equation (8) is crucial because we will show the polynomials $M_{r-k}^k(t)$ are generating functions for sets of rigged configurations.

Lemma 3.1. *The coefficient of* q^k *in* $K_{\lambda 2^r 1^{n-2r}}(q, t)$ *is*

$$\sum_{s=0}^{r} (-1)^{k-(r-s)} t^{(k-r+s)(n-r-s+1)+\binom{k-(r-s)}{2}} \begin{bmatrix} s \\ r-k \end{bmatrix}_t \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t). \tag{7}$$

Proof. Lemma 3.1 is a consequence of the q-binomial theorem [And, 3.3.6]. □

Definition 3.1. *Define the polynomials* $M_m^d(t)$ *recursively by* $M_m^0(t) = K_{\lambda 2^m 1^{n-2m}}(t)$ *and* $M_m^{d+1}(t) = M_m^d(t) - t^{n-2m-(d+1)} M_{m+1}^d(t)$.

Lemma 3.2. *The coefficient of q^k in $K_{\lambda 2^r 1^{n-2r}}(q, t)$ is $\left[{r \atop k}\right]_t M^k_{r-k}(t)$, so that*

$$K_{\lambda 2^r 1^{n-2r}}(q, t) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_t M^k_{r-k}(t) q^k. \tag{8}$$

Proof. We use induction on $r+k$ and (7) to show that

$$M^{k+1}_{r-(k+1)}(t) = \begin{bmatrix} r \\ k+1 \end{bmatrix}_t^{-1} \text{(coeff. of } q^{k+1} \text{ in } K_{\lambda 2^r 1^{n-2r}}(q, t)).$$

The lemma is true if $r = 0$ or $k = 0$. Assume

$$M^a_{b-a} = \begin{bmatrix} b \\ a \end{bmatrix}_t^{-1} \text{(coeff. of } q^a \text{ in } K_{\lambda 2^b 1^{n-2b}}(q, t))$$

if a and b are nonnegative integers such that $a + b \leq r + k$.

$$\begin{aligned}
M^{k+1}_{r-(k+1)}(t) =& M^{k+1}_{(r-1)-k}(t) \\
=& M^{k}_{(r-1)-k}(t) - t^{n-2(r-1-k)-(k+1)} M^{k}_{(r-1)-k+1}(t) \\
=& \begin{bmatrix} r-1 \\ k \end{bmatrix}_t^{-1} \text{(coeff. of } q^k \text{ in } K_{\lambda 2^{(r-1)} 1^{n-2(r-1)}}(q, t)) \\
& - t^{n-2(r-1-k)-(k+1)} \begin{bmatrix} r \\ k \end{bmatrix}_t^{-1} \text{(coeff. of } q^k \text{ in } K_{\lambda 2^r 1^{n-2r}}(q, t)) \\
=& \begin{bmatrix} r-1 \\ k \end{bmatrix}_t^{-1} \sum_{s=0}^{r-1} (-1)^{k-(r-1-s)} t^{(k-(r-1)+s)(n-(r-1)-s+1)+\binom{k-(r-1-s)}{2}} \\
& \times \begin{bmatrix} s \\ r-1-k \end{bmatrix}_t \begin{bmatrix} r-1 \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t) \\
& - t^{n-2(r-1-k)-(k+1)} \begin{bmatrix} r \\ k \end{bmatrix}_t^{-1} \sum_{s=0}^{r} (-1)^{k-(r-s)} t^{(k-r+s)(n-r-s+1)+\binom{k-(r-s)}{2}} \\
& \times \begin{bmatrix} s \\ r-k \end{bmatrix}_t \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t).
\end{aligned}$$

We will finish proving the lemma assuming $0 \leq s \leq r-1$. The case $s = r$ is a degenerate special case of what follows. Now we need to show the coefficient of $K_{\lambda 2^s 1^{n-2s}}(t)$, $0 \leq s \leq r-1$, in the last expression in this string of equalities is equal to

$$(-1)^{k+1-r+s} t^{(k+1-r+s)(n-r-s+1)+\binom{k+1-(r-s)}{2}} \begin{bmatrix} r \\ k+1 \end{bmatrix}_t^{-1} \begin{bmatrix} s \\ r-1-k \end{bmatrix}_t \begin{bmatrix} r \\ s \end{bmatrix}_t,$$

which is the coefficient of $K_{\lambda 2^s 1^{n-2s}}(t)$ in $\left[{r \atop k+1}\right]_t^{-1}$(coeff. of q^{k+1} in $K_{\lambda 2^r 1^{n-2r}}(q,t)$).

The coefficient of $K_{\lambda 2^s 1^{n-2s}}(t)$ in the last expression in the string of equalities is equal to

$$(-1)^{k+1-r+s} t^{(k+1-r+s)(n-r-s+1)+\binom{k+1-(r-s)}{2}} \times \left(\begin{bmatrix} r-1 \\ k \end{bmatrix}_t^{-1} t^{k+1-r+s} \begin{bmatrix} s \\ r-1-k \end{bmatrix}_t \begin{bmatrix} r-1 \\ s \end{bmatrix}_t + t^{(n-2(r-1-k)-(k+1))-(n-r-s+1)-(k-r+s)} \begin{bmatrix} r \\ k \end{bmatrix}_t^{-1} \begin{bmatrix} s \\ r-k \end{bmatrix}_t \begin{bmatrix} r \\ s \end{bmatrix}_t \right).$$

Using the q-factorial definition of the q-binomial coefficient, this turns into

$$(-1)^{k+1-r+s} t^{(k+1-r+s)(n-r-s+1)+\binom{k+1-(r-s)}{2}} \begin{bmatrix} r \\ k+1 \end{bmatrix}_t^{-1} \begin{bmatrix} s \\ r-1-k \end{bmatrix}_t \begin{bmatrix} r \\ s \end{bmatrix}_t \times \left(\frac{r_t}{(k+1)_t} t^{k+1-r+s} \times \frac{(r-s)_t}{r_t} + \frac{(r-k)_t}{(k+1)_t} \times \frac{(s-r+k+1)_t}{(r-k)_t} \right),$$

where $n_t = 1 - t^n$. The quantity in the brackets boils down to one, so we are done. □

4. Difference polynomials are nonnegative

In this section, we show that the polynomials $M_m^k(t)$, defined in the last section, are the generating functions for sets of rigged configurations, thus showing that $K_{\lambda 2^r 1^{n-2r}}(q, t)$ has nonnegative coefficients.

Definition 4.1. *Let* $\mathscr{M}_m^k = \{(\alpha(m), L) \in \mathscr{M}_m^0 | L^1_{\alpha_2^1+1} = L^1_{\alpha_2^1+2} = \cdots = L^1_{\alpha_2^1+k} = 0\}$. *Note that if* $\alpha_1^1 - \alpha_2^1 < k$, *then* $(\alpha(m), L) \notin \mathscr{M}_m^k$ *for any* L.

Lemma 4.1. *The generating function for the set* $\mathscr{M}_m^k$ *is* $M_m^k(t)$; *that is,*

$$\sum_{(\alpha(m), L) \in \mathscr{M}_m^k} t^{c((\alpha(m), L))} = M_m^k(t).$$

Proof. The proof is by induction. The lemma is true if $k = 0$, by the orginal Kirillov and Reshetikhin result. The definition of $M_m^{k+1}(t)$ is

$$M_m^k(t) - t^{n-2m-(k+1)} M_{m+1}^k(t).$$

In order to prove the lemma we need an injection $\phi_k : \mathscr{M}_{m+1}^k \to \mathscr{M}_m^k$ such that

(1) $c(\phi_k((\alpha(m+1), L))) = c((\alpha(m+1), L)) + n - 2m - (k+1)$ and
(2) $(\alpha(m), L) \in \mathscr{M}_m^k$ is not in the image of ϕ_k if and only if $(\alpha(m), L) \in \mathscr{M}_m^{k+1}$.

Let $\phi_k(\alpha(m+1), L) = (\alpha(m), \hat{L})$, where

$$\hat{L}_j^i = \begin{cases} L_j^i + 1 & \text{if } i = 1 \text{ and } j \geq \alpha_2^1 + k + 1, \\ L_j^i & \text{otherwise.} \end{cases}$$

Please note $(\alpha(m), \hat{L}) \in \mathscr{M}_m^k$ by (6) and also that (2) above is satisfied.

To see that (1) above is satisfied,

$$c((\alpha(m+1), L)) = \binom{n-(m+1)}{2} + \binom{m+1}{2} - (n-(m+1))\alpha_1^1 - (m+1)\alpha_2^1 + c'(\alpha) + \sum_{i,j\geq 1} L_j^i$$

and

$$\begin{aligned} &c(\phi_k(\alpha(m+1), L)) \\ &= c(\alpha(m), \hat{L}) \\ &= \binom{n-m}{2} + \binom{m}{2} - (n-m)\alpha_1^1 - m\alpha_2^1 + c'(\alpha) + \sum_{i,j\geq 1} \hat{L}_j^i \\ &= \binom{n-m}{2} + \binom{m}{2} - (n-m)\alpha_1^1 - m\alpha_2^1 + c'(\alpha) + \sum_{i,j\geq 1} L_j^i + (\alpha_1^1 - (\alpha_2^1 + k)) \\ &= \binom{n-(m+1)}{2} + n - m + \binom{m+1}{2} - (m+1) - (n-(m+1))\alpha_1^1 \\ &\quad - (m+1)\alpha_2^1 + \sum_{i,j\geq 1} L_j^i - k \\ &= c(\alpha(m+1), L) + n - 2m - k - 1. \quad \square \end{aligned}$$

The following formula is a consequence of Lemma 3.1 and Lemma 3.2:

$$M_m^k(t) = t^{\binom{n-m}{2}+\binom{m}{2}} \sum_{j=0}^{k} (-1)^j t^{-\binom{j}{2}-\binom{n-(m+j)}{2}-\binom{m+j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{t^{-1}} K_{\lambda 2^{m+j} 1^{n-2(m+j)}}(t).$$

Lemma 4.1 therefore has the following corollary.

Corollary 4.1. *The polynomial*

$$t^{\binom{n-m}{2}+\binom{m}{2}} \sum_{j=0}^{k} (-1)^j t^{-\binom{j}{2}-\binom{n-(m+j)}{2}-\binom{m+j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{t^{-1}} K_{\lambda 2^{m+j} 1^{n-2(m+j)}}(t)$$

has nonnegative coefficients.

Let $\tilde{K}_{\lambda\mu}(t)$ be the cocharge polynomial; that is, $\tilde{K}_{\lambda\mu}(t) = t^{n(\mu)} K_{\lambda\mu}(t^{-1})$, where $n(\mu) = \sum (i-1)\mu_i$. Then we rewrite the sum in Corollary 4.1 in terms of cocharge polynomials and Lemma 4.1 has a second corollary.

Corollary 4.2. *The polynomial*

$$\sum_{j=0}^{k} (-1)^j t^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_t \tilde{K}_{\lambda 2^{m+j} 1^{n-2(m+j)}}(t)$$

has nonnegative coefficients.

5. Proof of Theorem 1.1

In this section we finish proving the main theorem of this paper. We define the statistics c_r and cut_r on $\mathcal{M}_0^0$, the standard rigged configurations. Then we construct a surjection from $\mathcal{M}_0^0$ onto $\bigcup_{k=0}^{r} \mathcal{M}_{r-k}^{k}$ which respects the statistics.

Definition 5.1. *Let* $(\alpha(0), L) \in \mathcal{M}_0^0$ *and let* $a_i = L^1_{\alpha_2^1+i}$, *that is,* a_i *is the label of the* i*th column of length* 1 *in* $(\alpha(0), L)$. *Let* $a_{\alpha_1^1-\alpha_2^1+1} = \infty$ *and let* $a_0 = 0$. *We define* $cut_r(\alpha(0), L)$ *to be the least* j, $0 \le j \le r$, *such that* $j + 1 + a_{j+1} > r$. *If there is no such* j, *we let* $cut_r(\alpha(0), L) = r$. *Note that if* $\alpha_1^1 - \alpha_2^1 \le r$, *then* $cut_r(\alpha(0), L) \le \alpha_1^1 - \alpha_2^1$.

Definition 5.2. *Let* $(\alpha(0), L) \in \mathcal{M}_0^0$. *Let* $k = cut_r(\alpha(0), L)$. *Then we define* $c_r((\alpha(0), L)) = c((\alpha(0), L)) - (n-r)(r-k)$.

Definition 5.3. *Let* $k = cut_r(\alpha(0), L)$. *Define* $\Psi((\alpha(0), L)) = (\alpha(r-k), \tilde{L})$, *where*

$$\tilde{L}_j^i = \begin{cases} 0, & i = 1 \text{ and } \alpha_2^1 + 1 \le j \le \alpha_2^1 + k, \\ L_j^i - (r-k), & i = 1 \text{ and } \alpha_2^1 + k + 1 \le j \le \alpha_1^1, \\ L_j^i, & \text{otherwise.} \end{cases}$$

Since $P_1^1(\alpha(m+1)) = P_1^1(\alpha(m)) - 1$, if $L_j^1 \le P_1^1(\alpha(0))$, then $\tilde{L}_j^1 \le P_1^1(\alpha(r-k))$, so that $\Psi(\alpha(0), L) \in \mathcal{M}_{r-k}^0$. Since $\tilde{L}_j^1 = 0$ for $\alpha_2^1 + 1 \le j \le \alpha_2^1 + k$, $\Psi((\alpha(0), L)) \in \mathcal{M}_{r-k}^k$. Also note that the image of Ψ is $\bigcup_{j=0}^{r} \mathcal{M}_{r-j}^{j}$.

Lemma 5.1. *Let* $(\alpha(r-k), \tilde{L}) \in \mathcal{M}_{r-k}^k$. *Then*

$$\sum_{(\alpha(0), L) \in \Psi^{-1}((\alpha(r-k), \tilde{L}))} t^{c_r((\alpha(0), L))} = \begin{bmatrix} r \\ k \end{bmatrix}_t t^{c((\alpha(r-k), \tilde{L}))}.$$

Proof. First we show that

$$c(\Psi((\alpha(0), L))) = c(\alpha(0), L) - \left(\sum_{i=\alpha_2^1+1}^{\alpha_2^1+k} L_i^1 \right) - (n-r)(r-k)$$

so that

$$\begin{aligned} c_r(\alpha(0), L) &= \sum_{i=\alpha_2^1+1}^{\alpha_2^1+k} L_i^1 + c(\Psi(\alpha(0), L)) \\ &= \sum_{i=\alpha_2^1+1}^{\alpha_2^1+k} L_i^1 + c(\alpha(r-k), \tilde{L}), \end{aligned} \tag{9}$$

$$\begin{aligned}
&c(\Psi((\alpha(0), L)))\\
&= \sum_{i=k+1+\alpha_2^1}^{\alpha_1^1} L_i^1 - (\alpha_1^1 - \alpha_2^1 - k)(r-k) + \sum_{i>1, j\geq 1} L_j^i + c(\alpha(r-k))\\
&= \sum_{i\geq 1, j\geq 1} L_j^i - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 - (\alpha_1^1 - \alpha_2^1 - k)(r-k)\\
&\quad + \left[\binom{n-(r-k)}{2} + \binom{r-k}{2} - (n-(r-k))\alpha_1^1 - (r-k)\alpha_2^1 + c'(\alpha)\right] \text{ by (5)}\\
&= \sum_{j, i\geq 1} L_j^i - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 + \binom{n-(r-k)}{2} + \binom{r-k}{2} + n\alpha_1^1 + c'(\alpha) + k(r-k)\\
&= c(\alpha(0), L) - \binom{n}{2} - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 + \binom{n-(r-k)}{2} + \binom{r-k}{2} + k(r-k)\\
&= c(\alpha(0), L) - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 - (n-r)(r-k).
\end{aligned}$$

Also note that $\psi^{-1}((\alpha(r-k), \widetilde{L})) = \{(\alpha(0), L)\}$ such that

$$L_j^i \begin{cases} \leq r-k & \text{for } \alpha_2^1 + 1 \leq j \leq \alpha_2^1 + k \text{ and } i = 1, \\ = \widetilde{L}_j^1 + (r-k) & \text{for } j \geq \alpha_2^1 + k + 1 \text{ and } i = 1, \\ = \widetilde{L}_j^i & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned}
&\sum_{(\alpha(0), L)\in\Psi^{-1}(\alpha(r-k), \tilde{L})} t^{c_r(\alpha(0), L)}\\
&\quad = \sum_{0\leq L^1_{\alpha_2^1+1}\leq\cdots\leq L^1_{\alpha_2^1+k}\leq r-k} t^{\sum_{j=1}^{k} L^1_{\alpha_2^1+j} + c(\alpha(r-k), \tilde{L})} \quad \text{by (9)}\\
&\quad = \begin{bmatrix} r \\ k \end{bmatrix}_t t^{c(\alpha(r-k), \tilde{L})}.
\end{aligned}$$

This last lemma, the fact that the image of Ψ is $\bigcup_{j=0}^{r} \mathscr{M}_{r-j}^{j}$, and Lemma 3.2 and Lemma 4.1, finish the proof of Theorem 1.1. □

Acknowledgments

The author would like to thank John Stembridge for suggesting this problem to her and the referee for so carefully reading her paper.

REFERENCES

[And] George E. Andrews, *The theory of partitions*, Addison-Wesley, Reading, MA, 1976.

[GH] Adriano M. Garsia and Mark Haiman, *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90** (1993), 3607–3610.

[KR1] A. N. Kirillov and N. Yu. Reshetikhin, *Combinatorics, Bethe ansatz, and representations of the symmetric group*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **155** (1986), 50–64; English transl., J. Soviet Math. **41** (1988), 916–924.

[KR2] ———, *The Bethe ansatz and the combinatorics of Young tableaux*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **155** (1986), 65–115; English transl., J. Soviet Math. **41** (1988), 925–955.

[Mac1] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press, Oxford, 1979.

[Mac2] ———, *A new class of symmetric functions*, Actes 20^e Séminaire Lotharingien (1988), 131–171.

[Ste] J. R. Stembridge, *Some particular entries of the two-parameter Kostka matrix*, Proc. Amer. Math. Soc. **121** (1994), 367–373.

DEPARTMENT OF MATHEMATICS, SOUTHERN CONNECTICUT STATE UNIVERSITY, NEW HAVEN, CONNECTICUT 06515

E-mail address: `fishel@scsud.ctstateu.edu`