ON A RECTANGLE INCLUSION PROBLEM

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Abstract. We show that if every set of reals of size \(2^{\aleph_0}\) contains a meager-to-one continuous image of a set that cannot be covered by less than \(2^{\aleph_0}\) meager sets, then there exists a null (Lebesgue measure zero) subset of the plane \(\mathbb{R} \times \mathbb{R}\) that meets every nonnull rectangle \(X \times Y\). The antecedent is satisfied, e.g., if \(\omega_2\) Cohen reals are added to a model of the continuum hypothesis.

Martin's Axiom implies that a conull (i.e., with null complement) subset of the Euclidean plane \(\mathbb{R} \times \mathbb{R}\) contains a nonnull rectangle \(X \times Y\). Fremlin [5], Problem AS (see also [6], 3K), asked if this is true in ZFC.

It is known that there exists a conull subset of \(\mathbb{R} \times \mathbb{R}\) which contains no rectangle \(X \times Y\) with one side nonnull and the other measurable and nonnull. Namely, let \(E = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y \in F\}\), where \(F = \mathbb{R} \setminus \mathbb{Q}\). Clearly, \(E\) is a conull subset of \(\mathbb{R} \times \mathbb{R}\). If \(X \times Y \subseteq E\), then \(X + Y \subseteq F\). But, by a theorem of Steinhaus (see [9]), if \(X\) is measurable nonnull and \(Y\) is nonnull, then \(X + Y\) has nonempty interior; hence \(X + Y\) cannot be contained in \(F\).

(However, Brodskij and Eggleston (see [4]) showed that a measurable nonnull subset of \(\mathbb{R} \times \mathbb{R}\) always contains a rectangle \(X \times Y\) with \(X\) perfect and \(Y\) measurable nonnull.)

Consider the following proposition.

\[\text{(+)}\] If an \(F^\sigma\) subset of \(\mathbb{R} \times \mathbb{R}\) contains a nonnull rectangle \(X \times Y\), then it contains a measurable nonnull rectangle \(A \times B\).

Proposition \((\text{+})\) implies that there exists a conull subset of \(\mathbb{R} \times \mathbb{R}\) which contains no nonnull rectangle. Any conull \(F^\sigma\) subset of the set \(E\) considered above will do.

Proposition \((\text{+})\) has other interesting consequences (see [1]). For instance, it follows from \((\text{+})\) that if \(X\) and \(Y\) are nonnull subsets of \(\mathbb{R}\), then \(X + Y\) is nonmeager, hence every meager subgroup of \(\mathbb{R}\) is null. (If \(X + Y\) is covered by an \(F^\sigma\) set \(F\), then \(F^* = \{(x, y) : x + y \in F\}\) is an \(F^\sigma\) cover for \(X \times Y\). By \((\text{+})\), \(F^*\) contains a closed nonnull rectangle \(A \times B\). By the theorem of Steinhaus mentioned above, \(A + B\) has nonempty interior. Hence, \(F\) has nonempty interior.)

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Consistency of (+) was shown independently by Friedman and Shelah (see [1] and [2]). The model used was \( \omega_2 \) Cohen reals over a model of the continuum hypothesis.

In this paper we show that (+) is implied by a combinatorial condition whose consistency has been known for some time.

Denote by \( \mathcal{P}(\mathbb{R}) \) the family of all subsets of \( \mathbb{R} \). Let \( \lambda \) be a cardinal number. For \( \mathcal{A} \subseteq \mathcal{P}(\mathbb{R}) \) and \( X \subseteq \mathbb{R} \) write \( X \in \mathcal{A}_\lambda \) if \( X \) can be covered by less than \( \lambda \) sets from \( \mathcal{A} \), otherwise write \( X \in \mathcal{A}^+ \). Write also \( X \in \mathcal{A}^+ \) if \( X \in \mathcal{A}^+_\lambda \).

Let \( \mathcal{M} \) be the \( \sigma \)-ideal of meager subsets of \( \mathbb{R} \). Say that a function is meager-to-one if preimages of points are meager.

For \( D \subseteq \mathbb{R} \times \mathbb{R} \) and \( y \in \mathbb{R} \) let \( D^y \) denote the horizontal section of \( D \) determined by \( y \), i.e., \( D^y = \{ x \in \mathbb{R} : (x, y) \in D \} \).

Consider the following condition.

\((*)_\lambda\) Every set of reals of size \( \lambda \) contains a meager-to-one continuous image of a set from \( \mathcal{M}^+ \).

Note that we do not weaken \((*)_\lambda\) if we replace ‘continuous’ by ‘Baire measurable’. This is because Baire measurable functions are continuous on comeager sets (see [9]).

It is folklore that \((*)_{2^{\aleph_0}}\) holds if \( \omega_2 \) Cohen reals are added to a model of the continuum hypothesis. For example, Miller [8], p. 577, showed that in this model every set of reals of size \( 2^{\aleph_0} \) contains a one-to-one continuous image of a \( (2^{\aleph_0}, \aleph_1) \)-Lusin set. (X \( \subseteq \mathbb{R} \) is a \( (\lambda, \kappa) \)-Lusin set if \( |X| = \lambda \) and \( |X \cap S| < \kappa \) for all meager \( S \).) This immediately gives \((*)_{2^{\aleph_0}}\).

We prove:

\textbf{Theorem.} \((*)_{2^{\aleph_0}} \Rightarrow (+)\).

The Theorem is a consequence of the following Lemma.

\textbf{Lemma.} Assume \((*)_{2^{\aleph_0}}\). Then, with every nonnull set \( X \subseteq \mathbb{R} \) we can associate a family \( \{ E_n : n < \omega \} \) of closed nonnull subsets of \( \mathbb{R} \) such that if \( X \subseteq \bigcup_{m<\omega} D_m \), \( D_m \subseteq \mathbb{R} \) closed sets, then for some \( m \) and \( n \), \( E_n \subseteq D_m \).

\textbf{Proof of the Theorem.} Suppose that \( X \times Y \subseteq \mathbb{R} \times \mathbb{R} \) is a nonnull rectangle covered by \( \bigcup_{m<\omega} F_m \), \( F_m \subseteq \mathbb{R} \times \mathbb{R} \) closed. Clearly \( X \) and \( Y \) are nonnull. Let the \( E_n \)'s be as in the Lemma. Let \( G_{nm} = \{ y : E_n \subseteq (F_m)^y \} \). Then \( E_n \times G_{nm} \subseteq F_m \).

Each \( G_{nm} \) is closed. By the Lemma, the \( G_{nm} \)'s cover \( Y \). Indeed, let \( y \in Y \). We have \( X \times \{ y \} \subseteq \bigcup_m F_m \). So, \( X \subseteq \bigcup_m (F_m)^y \); hence, by the Lemma, some \( (F_m)^y \) contains some \( E_n \), i.e. \( y \in G_{nm} \).

Since \( Y \) is nonnull, it follows that some \( G_{nm} \) is nonnull. Thus, we can take \( E_n \times G_{nm} \) as our measurable nonnull rectangle, which is contained in \( F_m \). \( \square \)

\textbf{Proof of the Lemma.} Assume \((*)_{2^{\aleph_0}}\). Fix a nonnull set \( X \subseteq \mathbb{R} \).

\textbf{Claim.} There is \( Y \in \mathcal{M}^+_{2^{\aleph_0}} \) and a meager-to-one continuous function \( f : Y \rightarrow X \) such that for each closed null set \( W \), \( f^{-1}[W] \in \mathcal{M}^+_{2^{\aleph_0}} \).

\textbf{Proof.} Let \( \{ E_\xi : \xi < 2^{\aleph_0} \} \) be an enumeration of all closed null sets. Pick inductively \( x_\xi \in X \setminus \{ \{ x_\xi : \xi < \xi \} \cup \bigcup_{\xi < \xi} E_\xi \} \) (\( \xi < 2^{\aleph_0} \)). This is possible because \( X \) is not null and \( \{ x_\xi : \xi < \xi \} \cup \bigcup_{\xi < \xi} E_\xi \) is null. (By \((*)_{2^{\aleph_0}}\), \( \mathbb{R} \in \mathcal{M}^+_{2^{\aleph_0}} \); by [7], Thm. 2.1, this implies that a union of less than \( 2^{\aleph_0} \) closed null sets is...
null.) By the construction, the $x_\xi$'s are distinct and for every closed null set $W$, $|W \cap \{x_\xi : \xi < 2^\omega_0\}| < 2^\omega_0$.

By $(*)_{2^\omega_0}$ there is $Y \in \mathcal{M}_{2^\omega_0}$ and a meager-to-one continuous function $f : Y \mapsto \{x_\xi : \xi < 2^\omega_0\}$. Clearly $f^{-1}[W \cap X] \in \mathcal{M}_{2^\omega_0}$, for each closed null $W$. □

Let $\mathcal{U}$ be a countable base for $\mathbb{R}$. Let $\{U_n : n < \omega\}$ be an enumeration of all $U \in \mathcal{U}$ with the property that $f[U \cap Y]$ is not null. Let $E_n = f[U_n \cap Y]$ ($n < \omega$).

To see that this works suppose that $X \subseteq \bigcup_{m<\omega} D_m$, $D_m \subseteq \mathbb{R}$ closed sets. Then $f^{-1}[D_m \cap X]$'s are relatively closed in $Y$ sets that cover $Y$.

Suppose that for each $m$ and $U \in \mathcal{U}$ with $U \cap Y \subseteq f^{-1}[D_m \cap X]$, $f[U \cap Y]$ is null. Then, by the claim, $U \cap Y \in \mathcal{M}_{2^\omega_0}$. It follows that $Y$ is a union of countably many sets from $\mathcal{M}_{2^\omega_0}$ and countably many nowhere dense sets. Hence $Y \in \mathcal{M}_{2^\omega_0}$, which is a contradiction.

Thus, for some $U \in \mathcal{U}$ with $U \cap Y \subseteq f^{-1}[D_m \cap X]$, $f[U \cap Y]$ is not null, so it must be one of the $E_n$'s. Clearly, $f[U \cap Y] \subseteq D_m$. □

We shall now generalize the Lemma and the Theorem. Fix cardinals $\kappa \leq \lambda < 2^\omega_0$, $cf(\kappa) > \omega$, and an arbitrary family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ of closed sets. Note that $\mathcal{F}_\kappa^+$ is a $\sigma$-ideal.

Definition. Let $X \subseteq \mathbb{R}$. Say that a family $\{F_n : n < \omega\}$ of closed sets is $\kappa$-dense (for $X$) if, whenever $X$ is covered by less than $\kappa$ closed sets, then some one of these closed sets covers some $F_n$. Say $\sigma$-dense for $\aleph_1$-dense.

Note. Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ be a $\sigma$-ideal. Every closed set from $\mathcal{F}^+$ (so also every superset of such a set) has a $\sigma$-dense family contained in $\mathcal{F}^+$. Indeed, suppose that $X \in \mathcal{F}^+$ is closed. Let $\mathcal{U}$ be a countable base for $\mathbb{R}$. Let $X^* = X \setminus \{U \in \mathcal{U} : U \cap X \in \mathcal{F}\}$. Then $X^*$ is closed and each $U \cap X^*$ ($U \in \mathcal{U}$) is either empty or belongs to $\mathcal{F}^+$. As a $\sigma$-dense family we can just take the collection of those sets $U \cap X^*$ ($U \in \mathcal{U}$) which belong to $\mathcal{F}^+$. If $X^* \subseteq \bigcup_{m<\omega} D_m$, $D_m \subseteq \mathbb{R}$ closed, then for some $m$, $X^* \cap D_m$ has nonempty interior relatively to $X^*$ (Baire's category theorem). So, for some $U \in \mathcal{U}$, $U \cap X^* \neq \emptyset$ and $U \cap X^* \subseteq D_m$. Then $U \cap X^* \subseteq D_m$, and, by the definition of $X^*$, $U \cap X^* \in \mathcal{F}^+$.

It also follows that if $X$ is arbitrary and we can find a family of closed sets $\{F_n : n < \omega\} \subseteq \mathcal{F}^+$ such that every $F_\sigma$ set covering $X$ contains some $F_n$, then $X$ has a $\sigma$-dense family contained in $\mathcal{F}^+$.

Lemma 1. Let $\{F_n : n < \omega\}$ be a $\kappa$-dense family for $X$. Suppose that $X \times Y \subseteq \bigcup_{\xi < \mu} D_\xi$, where $\mu < \kappa$ and $D_\xi \subseteq \mathbb{R} \times \mathbb{R}$ ($\xi < \mu$) are closed. Then there are $Y_{n\xi} \subseteq Y$ ($n < \omega$, $\xi < \mu$) such that $\bigcup_{n, \xi} Y_{n\xi} = Y$ and for all $n$ and $\xi$, $F_n \times Y_{n\xi} \subseteq D_\xi$.

Proof. Let $Y_{n\xi} = \{y \in Y : F_n \subseteq (D_\xi)^y\}$. Then $F_n \times Y_{n\xi} \subseteq D_\xi$. Also, given $y \in Y$, $(D_\xi)^y$ ($\xi < \mu$) cover $X$. So, by the definition of a dense family, some $F_n$ is contained in some $(D_\xi)^y$, i.e., $y \in Y_{n\xi}$. □

Corollary. Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ be arbitrary. Let $\mu < \kappa$, and let $D_\xi \subseteq \mathbb{R} \times \mathbb{R}$ ($\xi < \mu$) be closed. Suppose that $X \times Y \subseteq \bigcup_{\xi} D_\xi$, where $Y \in \mathcal{F}_\kappa$ and $X$ has a $\kappa$-dense
family contained in \( \mathcal{F}_k^+ \). Then there exist closed sets \( A \in \mathcal{F}_k^+ \) and \( B \in \mathcal{F}_k^+ \) such that \( A \times B \subseteq D_\xi \) for some \( \xi \).

**Proof.** Let \( \{ F_n : n < \omega \} \subseteq \mathcal{F}_k^+ \) be \( \kappa \)-dense for \( X \). If in Lemma 1, \( Y \in \mathcal{F}_k^+ \), then some \( Y_n \in \mathcal{F}_k^+ \). Set \( A = F_n \) and \( B = \overline{Y_n} \). \( \square \)

**Definition.** Let \( \mathcal{F}_k \) be the collection of sets \( X \subseteq \bigcup \mathcal{F} \) with the property that for any continuous function \( f : Y \rightarrow X, \ Y \subseteq \mathbb{R} \), there is \( W \in \mathcal{F}_k \) with \( Y \setminus f^{-1}[W] \in \mathcal{M}_k \). Note that \( \mathcal{F}_k \) is a \( \sigma \)-ideal, which extends \( \mathcal{M}_k \).

**Lemma 2.** If \( X \notin \mathcal{F}_k \), then \( X \) has a \( \kappa \)-dense family contained in \( \mathcal{F}_k^+ \).

**Proof.** Suppose \( X \notin \bigcup \mathcal{F} \), then \( \{ \{x\} \} \) for any \( x \in X \setminus \bigcup \mathcal{F} \) is a \( \kappa \)-dense family. So, let \( X \subseteq \bigcup \mathcal{F} \) and let \( f : Y \rightarrow X \ (Y \subseteq \mathbb{R}) \) be a continuous function such that \( \forall W \in \mathcal{F}_k \ Y \setminus f^{-1}[W] \in \mathcal{M}_k^+ \). Let \( \mathcal{U} \) be a countable base for \( \mathbb{R} \). Let \( V = \bigcup \{ U \in \mathcal{U} : f[U \cap Y] \in \mathcal{F}_k \} \). Then \( f[V \cap Y] \in \mathcal{F}_k \), so, by our assumption, \( Y \setminus V \in \mathcal{M}_k^+ \).

As a \( \kappa \)-dense family we take \( \{ f[U \cap Y] : U \in \mathcal{U} \text{ and } f[U \cap Y] \in \mathcal{F}_k^+ \} \). To see that this works let \( \mu < \kappa \) and suppose that \( X \) is covered by closed sets \( D_\xi \subseteq \mathbb{R} \ (\xi < \mu) \). Then the sets \( f^{-1}[D_\xi] \cap Y \) cover \( Y \) and are relatively closed in \( Y \). If \( \xi \) is such that for every \( U \in \mathcal{U} \) with \( U \cap Y \subseteq f^{-1}[D_\xi] \), \( f[U \cap Y] \in \mathcal{F}_k \), then \( f^{-1}[D_\xi] \cap Y \setminus V \) is nowhere dense. Since \( Y \setminus V \in \mathcal{M}_k^+ \), this cannot happen to every \( \xi \). Thus, we can find \( \xi \) and \( U \) with \( f[U \cap Y] \notin \mathcal{F}_k \). Clearly \( f[U \cap Y] \subseteq D_\xi \). \( \square \)

**Definition.** Say that a sequence \( (F_\xi : \xi < \lambda) \subseteq \mathcal{F} \) is \( \kappa \)-cofinal in \( \mathcal{F} \) if \( \forall F \in \mathcal{F}_k \exists \xi < \lambda \ F \subseteq \bigcup_{\xi < \lambda} F_\xi \).

**Lemma 3.** Assume \( (\ast)_\lambda \). If \( \mathcal{F} \) has a \( \kappa \)-cofinal sequence of length \( \lambda \), then \( \mathcal{F}_k \subseteq \mathcal{F} \).

**Proof.** Fix \( X \in \mathcal{F}_k^+ \), \( X \subseteq \bigcup \mathcal{F} \). Let \( (F_\xi : \xi < \lambda) \subseteq \mathcal{F} \) be a \( \kappa \)-cofinal sequence. Pick inductively \( x_\xi \in X \setminus \bigcup \{ F_\xi : \xi < \zeta \} \cup \bigcup_{\xi < \zeta} F_\xi \) (\( \zeta < \lambda \)). This is possible because \( X \in \mathcal{F}_k^+ \) and \( X \subseteq \bigcup \mathcal{F} \). By the construction, \( x_\xi \)'s are distinct. Also, for every \( W \in \mathcal{F}_k \) there is \( \xi < \lambda \) with \( W \subseteq \bigcup_{\xi < \zeta} F_\xi \). Hence \( |W \cap \{ x_\xi : \xi < \zeta \}| < \lambda \).

By \( (\ast)_\lambda \) there is \( Y \in \mathcal{M}_k^+ \) and a meager-to-one continuous function \( f : Y \hookrightarrow \{ x_\xi : \xi < \lambda \} \). Note that for every \( W \in \mathcal{F}_k \), \( f^{-1}[W \cap X] \in \mathcal{M}_\lambda \). Since \( Y \in \mathcal{M}_k^+ \) and \( \kappa \leq \lambda \), we have that \( Y \setminus f^{-1}[W \cap X] \notin \mathcal{M}_\lambda \). Thus \( f \) witnesses that \( X \notin \mathcal{F}_k \). \( \square \)

**Corollary.** Assume \( (\ast)_\lambda \). If \( \mathcal{F} \) has a \( \kappa \)-cofinal sequence of length \( \lambda \), then each \( X \in \mathcal{F}_k^+ \) has a \( \kappa \)-dense family contained in \( \mathcal{F}_k^+ \).

**Proof.** By Lemmas 2 and 3. \( \square \)

Combining the corollaries of Lemmas 1 and 3 we get the following.

**Proposition 1.** Let \( \kappa < \lambda < 2^{\aleph_0} \) be cardinals, \( \text{cf}(\kappa) > \omega \). Assume \( (\ast)_\lambda \). Let \( \mathcal{F} \subseteq \mathcal{P}(\mathbb{R}) \) be arbitrary, and let \( \mathcal{F} \subseteq \mathcal{P}(\mathbb{R}) \) be a family of closed sets which has a \( \kappa \)-cofinal sequence of length \( \lambda \). Let \( \mu < \kappa \) and suppose that \( X \times Y \subseteq \bigcup_0 D_\xi \), where \( D_\xi \subseteq \mathbb{R} \times \mathbb{R} \ (\xi < \mu) \) are closed and \( X \in \mathcal{F}_k^+ \) and \( Y \in \mathcal{F}_k^+ \).
Then there are closed sets \( A \in \mathcal{F}_k^+ \) and \( B \in \mathcal{F}_k^+ \) such that \( A \times B \subseteq D_\xi \) for some \( \xi \).

To make it more transparent that Proposition 1 generalizes the Theorem recall the following notation (see [3]). Let \( \mathcal{F} \subseteq \mathcal{P}(\mathbb{R}) \) be such that \( \bigcup \mathcal{F} = \mathbb{R} \notin \mathcal{F} \).

\[
\begin{align*}
\text{cof}(\mathcal{F}) &= \min\{\mid \mathcal{A} \mid : \mathcal{A} \subseteq \mathcal{F}, \forall B \in \mathcal{F} \exists A \in \mathcal{A}, B \subseteq A\}; \\
\text{cov}(\mathcal{F}) &= \min\{\mid \mathcal{A} \mid : \mathcal{A} \subseteq \mathcal{F}, \bigcup \mathcal{A} = \mathbb{R}\}; \\
\text{add}(\mathcal{F}) &= \min\{\mid \mathcal{A} \mid : \mathcal{A} \subseteq \mathcal{F}, \bigcup \mathcal{A} \notin \mathcal{F}\}.
\end{align*}
\]

It is folklore that \( \text{add}(\mathcal{F}) \leq \text{cov}(\mathcal{F}) \leq \text{cof}(\mathcal{F}) \) and \( \text{add}(\mathcal{F}) \leq \text{cf}(\text{cof}(\mathcal{F})) \).

Let now \( \mathcal{F} \) be the family of all closed null subsets of \( \mathbb{R} \), and \( \mathcal{N} \) the \( \sigma \)-ideal of null subsets of \( \mathbb{R} \).

**Lemma 4.**

(a) \( \text{cov}(\mathcal{N}) \leq \text{cof}(\mathcal{N}) \) and \( (*) \lambda \Rightarrow \lambda \leq \text{cov}(\mathcal{N}) \) (so \( (*)_{\text{cof}(\mathcal{N})} \Rightarrow \text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N}) \));

(b) \( \mathcal{F}_{\text{cov}(\mathcal{N})} \subseteq \mathcal{N} \);

(c) \( \text{cof}(\mathcal{F}) \leq \text{cof}(\mathcal{N}) \);

(d) \( \text{add}(\mathcal{N}) \) is a regular cardinal such that \( \mathcal{N}_{\text{add}(\mathcal{N})} \subseteq \mathcal{N} \) and \( \aleph_0 < \text{add}(\mathcal{N}) \leq \text{add}(\mathcal{N}) \leq \text{cf}(\text{cof}(\mathcal{N})) \leq \text{cof}(\mathcal{N}) \).

**Proof.** (a) is trivial; (d) is well known from Cichoń's diagram (see [3]); (b) is a version of Thm. 2.1 of [7]. We sketch a proof of (c), which is a folklore heir to Thm. 2.1 of [7].

Let \( \lambda = \text{cof}(\mathcal{N}) \). There exist dominating and diagonalizing families of size \( \lambda \), i.e., \( \{f_\xi : \xi < \lambda\} \subseteq \omega_\omega \) and \( \{g_\eta : \eta < \lambda\} \subseteq \omega_\omega \) such that

\[
\begin{align*}
\forall e &\in \omega_\omega \exists \xi \forall n \; e(n) < f_\xi(n); \\
\forall h &\in \omega_\omega \exists \eta \forall n \exists m > n \; h(m) = g_\eta(m)
\end{align*}
\]

(this is because \( \omega \leq \text{cof}(\mathcal{N}) \) and \( \text{non}(\mathcal{N}) \leq \text{cof}(\mathcal{N}) \) in Cichoń's diagram; see [3]). We can assume without loss of generality that \( \forall \xi \forall n \; f_\xi(n) > n \).

Let \( \langle I^n : i < \omega \rangle \) be an enumeration of all finite unions of closed intervals with rational endpoints with measure \( \leq 2^{-n} \). Then the sets

\[
F_{\eta\xi} = \bigcap_n \bigcup_m \{I^n_{g_\eta(m)} : f^n_\xi(0) \leq m < f^{n+1}_\xi(0)\}
\]

are closed null and every closed null set is covered by some \( F_{\eta\xi} \) (here, \( f^n_\xi = f_\xi \circ f_\xi \circ \cdots \circ f_\xi \), \( n \) times).

Indeed, suppose that \( F \) is closed null. Then there is \( h \in \omega_\omega \) such that \( F \subseteq \bigcap_m I^m_{h(m)} \). Let \( \eta \) be such that \( \forall n \exists m > n \; h(m) = g_\eta(m) \). Define \( e \in \omega_\omega \) by \( e(n) = \min\{m \geq n : h(m) = g_\eta(m)\} \). Let \( \xi \) be such that \( \forall n \; e(n) < f_\xi(n) \).

Then \( \forall n \; e(f^n_\xi(0)) < f^{n+1}_\xi(0) \), so \( \forall n \exists m \in [f^n_\xi(0), f^{n+1}_\xi(0)) \; h(m) = g_\eta(m) \). It follows that \( \bigcap_m I^m_{h(m)} \subseteq F_{\eta\xi} \).

From Proposition 1 we get the following.

**Proposition 2.** Assume \( (*)_{\text{cof}(\mathcal{N})} \). If less than \( \text{add}(\mathcal{N}) \) closed subsets of \( \mathbb{R} \times \mathbb{R} \) cover a nonnull rectangle \( X \times Y \), then some of them cover a closed nonnull rectangle \( A \times B \).
Proof. Use Lemma 4. Let \( \lambda = \text{cof}(A) \), \( \kappa = \text{add}(\mathcal{N}) \). By (d), \( \aleph_0 < \text{cf}(\kappa) \) and \( \kappa \leq \lambda \). By (a), \( \lambda = \text{cov}(A) \). So, by (b), \( X \in \mathcal{K}_\lambda^+ \); and by \( \mathcal{K}_\kappa \subseteq \mathcal{N}, \ Y \in \mathcal{N}_\kappa^+ \). Also by (c) and \( \kappa \leq \text{cf}(\lambda) \), \( \mathcal{F} \) has a \( \kappa \)-cofinal sequence of length \( \lambda \). Now use Proposition 1 for \( \mathcal{F} = \mathcal{N} \). \( \square \)

Note that by Lemma 4(a), \( (*)_{2^{\omega_1}} \Rightarrow \text{cov}(A) = \text{cof}(A) = 2^{\aleph_0} \). So, Proposition 2 directly generalizes the Theorem.

We conclude the paper with the following variant of Proposition 2.

**Proposition 3.** Assume \( (*)_{\text{cof}(A)} \). Suppose that a coanalytic set \( C \subseteq \mathbb{R} \times \mathbb{R} \), whose all horizontal sections are unions of less than \( \text{cf}(\text{cof}(A)) \) closed sets, contains a nonnull rectangle \( X \times Y \). Then it contains a closed nonnull rectangle \( A \times B \).

**Proof.** Let \( \lambda = \text{cof}(A) \), \( \kappa = \text{cf}(\lambda) \). As in the proof of Proposition 2 we get that \( X \in \mathcal{K}_\lambda^+ \) and that there is a \( \kappa \)-cofinal sequence for \( \mathcal{F} \) of length \( \lambda \). So, by the corollary to Lemma 3, \( X \) has a \( \kappa \)-dense family \( \{F_n : n < \omega\} \subseteq \mathcal{K}_\kappa^+ \).

Let \( G_n = \{y : F_n \subseteq C^y\} \). Note that \( G_n \)'s are analytic. Also, they cover \( Y \) (\( C^y \) is a union of less than \( \kappa \) closed sets, so, by \( \kappa \)-density, \( C^y \) contains some \( F_n \)). As \( Y \) is nonnull, some \( G_n \) is nonnull and, hence, being measurable, contains a closed nonnull subset. Since \( F_n \) is also closed nonnull and \( F_n \times G_n \subseteq C \), we are done. \( \square \)

**Note.** Our rectangles had sides parallel to the coordinate axes. What if we consider arbitrary rectangles. The sets \( E \) and \( F \) discussed at the beginning of the paper have a stronger property than stated. Namely, if \( E \) contains a rectangle \( R \) with one side nonnull and the other measurable and nonnull, then one of the sides must be parallel to the line \( y = -x \). Indeed, otherwise \( R \) is obtained by a rotation by an angle \( \alpha, 0 \leq \alpha < \pi/4 \), of a rectangle \( X \times Y \). So, \( R = \{(x^*, y^*) : x \in X, y \in Y\} \), where \( x^* = x \cos \alpha - y \sin \alpha \) and \( y^* = x \sin \alpha + y \cos \alpha \). From \( R \subseteq E \) we get that for all \( x \in X \) and \( y \in Y \), \( x^* + y^* \in F \), i.e., \( x(\cos \alpha + \sin \alpha) + y(\cos \alpha - \sin \alpha) \in F \). Thus, \( a \cdot X + b \cdot Y \subseteq F \) for some nonzero \( a \) and \( b \). As before, if one of \( X, Y \) is nonnull and the other is measurable and nonnull, we get a contradiction.

Now let \( E' \) be a rotation of \( E \) by an angle \( \beta, 0 < \beta < \pi \). Then \( E \cap E' \) is a conull subset of the Euclidean plane, which contains no rectangle with one side nonnull and the other measurable and nonnull. Thus, from the following weaker version of \( (+) \):

\( (+') \) if an \( F_\omega \) subset of \( \mathbb{R} \times \mathbb{R} \) contains a nonnull rectangle, then it contains a measurable nonnull rectangle;

we can construct a conull subset of the plane which contains no nonnull rectangle.

**References**


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