TRANSFERENCE OF MAXIMAL MULTIPLIERS
ON HARDY SPACES

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Abstract. Based on the atomic decomposition of the Hardy space, we give a simple proof for a theorem of Liu and Lu (Studia Math. 105 (1993), 121–134), which discusses the relation between the maximal operators on \( \mathbb{R}^n \) and on \( T^n \). More significantly, our proof shows that condition (1) in Liu and Lu's Theorem 1 is superfluous.

1. INTRODUCTION

Let \( H^p(\mathbb{R}^n) \), \( 0 < p < \infty \), be the Hardy spaces defined by [FS]

\[
H^p(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n), \| \Phi^+ f \|_{L^p(\mathbb{R}^n)} < \infty \},
\]

where \( \Phi^+ f(x) = \sup_{t>0} |\Phi_t \ast f(x)| \), \( \Phi_t(x) = t^{-n} \Phi(x/t) \), and \( \Phi \in \mathcal{S}'(\mathbb{R}^n) \) is a radial function satisfying \( \int \Phi = 1 \). The corresponding periodic Hardy spaces are \( H^p(T^n) = \{ f \in \mathcal{S}'(T^n), \| \Phi^+ f \|_{L^p(T^n)} < \infty \} \), where \( \Phi^+ f(x) = \sup_{t>0} |\Phi_t \ast f(x)| \), \( \Phi_t(x) = \sum_{k \in \Lambda} \Phi(tk)e^{2\pi ik \cdot x} = C t^{-n} \sum_{k \in \Lambda} \Phi((x+k)/t) \) and \( \Lambda \) is the unit lattice which is the additive group of points in \( \mathbb{R}^n \) having integral coordinates.

Let \( \lambda \) be a bounded continuous function on \( \mathbb{R}^n \). For each \( \varepsilon > 0 \), define

\[
(T_\varepsilon f)^\sim(u) = \lambda(\varepsilon u)\hat{f}(u), \quad f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n),
\]

and

\[
\tilde{T}_\varepsilon f(x) = \sum_{k \in \Lambda} \lambda(\varepsilon k) a_k(f)e^{2\pi ik \cdot x}, \quad f \in L^2(T^n) \cap H^p(T^n).
\]

We say that \( \lambda \) is a maximal multiplier on \( H^p(\mathbb{R}^n) \) if \( T_\varepsilon f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)| \) can be extended to a bounded operator from \( H^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \). Similarly, \( \lambda \) is called a maximal multiplier on \( H^p(T^n) \) if \( \tilde{T}_\varepsilon f(x) = \sup_{\varepsilon>0} |\tilde{T}_\varepsilon f(x)| \) can be extended to a bounded operator from \( H^p(T^n) \) to \( L^p(T^n) \).

The relation between the maximal multipliers \( T^* \) and \( \tilde{T}^* \) was first studied by Kenig and Tomas when \( p > 1 \). Their result can be extended to the Lorentz
space $L(p, q)$, $p > 1$ (see [F]). Recently, Liu and Lu [LL] studied the case of $0 < p \leq 1$. Their main result is the following theorem.

**Theorem 1 [LL].** Let $0 < p \leq 1$, and let $\lambda$ be a bounded and continuous function on $\mathbb{R}^n$.

(i) Suppose that $\lambda$ is a maximal multiplier on $H^p(\mathbb{R}^n)$ such that

$$\lim_{|x| \to \infty} \lambda(x) = \alpha$$

exists. Then $\lambda$ is a maximal multiplier on $H^p(\mathbb{T}^n)$.

(ii) If $\lambda$ is a maximal multiplier on $H^p(\mathbb{T}^n)$, then $\lambda$ is a maximal multiplier on $H^p(\mathbb{R}^n)$.

Since the above condition (1) plays an important role in their proof, Liu and Lu asked (see p. 133 in [LL]) if condition (1) can be weakened any further.

In this note, we will show that condition (1) in Theorem 1 is superfluous. Based on the atomic decomposition of the Hardy space, our proof is much shorter and more direct than those in [LL]. The following is our main result.

**Theorem 2.** Let $\lambda$ be a continuous and bounded function on $\mathbb{R}^n$, $0 < p \leq 1$. If

$$\|T^* f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{H^p(\mathbb{R}^n)}$$

for all $f \in H^p(\mathbb{R}^n)$,

then

$$\|\tilde{T}^* \tilde{f}\|_{L^p(\mathbb{T}^n)} \leq C \|\tilde{f}\|_{H^p(\mathbb{T}^n)}$$

for all $\tilde{f} \in H^p(\mathbb{T}^n)$.

For the sake of simplicity, $C$ always denotes a positive constant which may vary at each of its occurrences.

To prove Theorem 2, we need to use the atomic characterization of the Hardy space. A regular $(p, 2, s)$ atom is a function $a(x)$ supported in some ball $B(x_0, \rho)$ satisfying

(i) $\|a\|_2 \leq \rho^{-n/p + n/2};$

(ii) $\int_{\mathbb{R}^n} a(x)P(x) \, dx = 0$

for all polynomials $P(x)$ of degree less than or equal to $s$.

The space $H_{p, s}^p(\mathbb{R}^n)$, $0 < p \leq 1$, is the space of all distributions $f \in \mathcal{S}^\prime(\mathbb{R}^n)$ having the form

$$f = \sum c_k a_k$$

and satisfying

$$\sum |c_k|^p < \infty,$$

where each $a_k$ is a $(p, 2, s)$ atom. The "norm" $\|f\|_{H_{p, s}^p(\mathbb{R}^n)}$ is the infimum of all expressions $(\sum |c_k|^p)^{1/p}$ for which we have a representation (2) of $f$. A well-known fact (see [FoSi]) is that $\|f\|_{H_{p, s}^p(\mathbb{R}^n)} \cong \|f\|_{H^p(\mathbb{R}^n)}$, and in particular, $\|a\|_{H^p(\mathbb{R}^n)} \leq C$, with a constant $C$ independent of the $(p, 2, s)$ atom $a(x)$ if $s \geq \lceil n(1/p - 1) \rceil$.

We also have a similar decomposition theorem for any function $g \in H^p(\mathbb{T}^n)$. In particular, suppose $g \in H^p(\mathbb{T}^n) \cap \mathcal{S}(\mathbb{T}^n)$ and its Fourier coefficient

$$a_0(g) = \int_{\mathbb{T}^n} g(x) \, dx = 0,$$
where \( Q = \{ x \in \mathbb{R}^n : -1/2 \leq x_j < 1/2, \ j = 1, 2, \ldots, n \} \) is the fundamental cube on which
\[
\int_{T^n} g(x) \, dx = \int_Q g(x) \, dx
\]
for all functions \( g \) on \( T^n \). Then we have the following lemma.

**Lemma 4.** Suppose \( g \in H^p(T^n) \cap \mathcal{S}(T^n) \) with \( a_0(g) = 0 \). If we restrict \( x \) to \( Q \), then for any fixed positive integer \( s \)
\[
g(x) = \sum c_k a_k(x),
\]
where each \( a_k(x) \) is a \((p, 2, s)\) atom satisfying \( a_k(x + n) = a_k(x) \) for \( n \in \Lambda \) and \( \|g\|_{H^p(T^n)} = \|\sum |c_k|^p \|^p \).

**Proof.** Choose a radial function \( \phi \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{supp}(\phi) \subset B(0, 1) \). In addition, we can choose such a \( \phi \) such that \( \int x^J \phi(x) \, dx = 0 \) for all multi-indices \( J, \ |J| \leq s \), and \( \int_0^\infty \phi(tx)^2 t^{-1} \, dt = 1 \) for all \( x \neq 0 \). We let \( \tilde{\phi}_t(x) = \sum_{t \in \Lambda \setminus \{0\}} \tilde{\phi}(tk)e^{2\pi ik \cdot x} = C \sum_{k \in \Lambda} t^{-n} \phi((x+k)/t) \). Then by checking the Fourier coefficients, we easily obtain the following Calderón reproducing formula:
\[
(5) \quad g(x) = \int_0^\infty \left( \phi_t * \phi_t * g \right)(x) t^{-1} \, dt = \int_0^1 + \int_0^\infty.
\]

Now by a standard argument [FoS] (or see [BF] for the proof on any compact Lie group), one can easily obtain that
\[
g(x) = \sum c_k a_k(x),
\]
where each \( a_k(x) \) is a \((p, 2, s)\) atom satisfying \( a_k(x + n) = a_k(x) \) for \( n \in \Lambda \) and \( \sum |c_k|^p \approx \|S_{\phi}(g)\|_{L^p(T^n)} \). Here \( S_{\phi}(g) \) is defined by
\[
S_{\phi}(g)(x) = \int_{|x-y|<t} |(g * \tilde{\phi}_t)(y)|^2 t^{-n-1} \, dy \, dt.
\]
So to prove the lemma it suffices to show that \( \|S_{\phi}(g)\|_{L^p(T^n)} \approx \|g\|_{H^p(T^n)} \) for all \( g \in H^p(T^n) \cap \mathcal{S}(T^n) \). But the proof for \( \|S_{\phi}(g)\|_{L^p(T^n)} \approx \|g\|_{H^p(T^n)} \) is, mutatis mutandis, the same as for \( \mathbb{R}^n \) (see [FoS]) without using any new techniques or ideas.

The following lemma is Lemma 3.1 in [F]. For completeness, we state its proof.

**Lemma 6.** Suppose that \( \Psi(x) \) is a continuous function with compact support. Let \( \lambda(x) \) be a bounded and continuous function on \( \mathbb{R}^n \), and let \( T_\varepsilon \) and \( \tilde{T}_\varepsilon \) be the families of operators on \( \mathbb{R}^n \) and \( T^n \), respectively, associated to the function \( \lambda \). Take \( \Psi^{1/N}(\xi) = \Psi(\xi/N) \). If \( \Psi \) satisfies \( \Psi(0) = 1 \) and \( \hat{\Psi} \in L^1(\mathbb{R}^n) \), then for any \( g \in \mathcal{S}(T^n) \) and any positive integer \( N \),
\[
(7) \quad \Psi(y/N)(\tilde{T}_\varepsilon g)(y) = T_\varepsilon(g\Psi^{1/N})(y) + J_{N,\varepsilon}(y)
\]
for all \( y \in \mathbb{R}^n \), where \( J_{N,\varepsilon}(y) \) tends to zero uniformly for \( y \in \mathbb{R}^n \) and \( 0 \leq \varepsilon \leq R \) \((R > 0 \text{ is any fixed number})\), as \( N \to \infty \).

**Proof.** Since \( g(x) = \sum a_k(g) e^{2\pi ik \cdot x} \) with the Fourier coefficients \( \{a_k(g)\} \) rapidly decreasing as \( |k| \to \infty \), it suffices to prove the lemma when \( g(x) = \).
\( e_k(x) = e^{2\pi i k \cdot x} \). In this case
\[
|J_{N, e}(y)| = |\Psi^{1/N}(y)(\tilde{T}_e e_k)(y) - T_e(\Psi^{1/N} e_k)(y)|
\]
\[
= \left| e_k(y) \int_{\mathbb{R}^n} N^n \tilde{\Phi}(N \xi) e^{2\pi i y \cdot \xi} \left\{ \lambda(e_k) - \lambda(e_k + e \xi) \right\} d\xi \right|
\]
\[
\leq \int_{\mathbb{R}^n} |\tilde{\Phi}(\xi)| \left| \frac{d}{d\xi} \right| \left\{ \lambda(e_k) - \lambda(e_k + N^{-1} e \xi) \right\} d\xi.
\]
Since \( \tilde{\Phi} \) is integrable and since \( \lambda \) is bounded and continuous, the last quantity converges to zero as \( N \to \infty \). The lemma is proved.

2. Proof of the main theorem

By a note on page 128 in [LL] we only need to show that for any \( g \in S(\mathbb{T}^n) \cap H^p(\mathbb{T}^n) \) with \( a_0(g) = 0 \),

\[
\| \tilde{T}^* g \|_{L^p(\mathbb{T}^n)} \leq C \| g \|_{H^p(\mathbb{T}^n)}.
\]

For any \( R > 0 \) fixed, we define \( \tilde{T}_R g(x) = \sup_{0 < \xi \leq R} |\tilde{T}_e g(x)| \). Since as \( R \to \infty \) \( \tilde{T}_R g(x) \) increases pointwise to \( \tilde{T}^* g(x) \), by monotone convergence theorem, to prove (8) we only need to prove that

\[
\| \tilde{T}_R g \|_{L^p(\mathbb{T}^n)} \leq C \| g \|_{H^p(\mathbb{T}^n)}
\]

with a constant \( C \) independent of \( R \) and \( g(x) \).

By Lemma 4, \( g(x) = \sum c_k a_k(x) \), where each \( a_k \) is a \([n(1/p - 1)] + 2n \) atom and \( \sum |c_k|^p \leq \| g \|_{H^p(\mathbb{T}^n)}^p \).

We let
\[
\Psi(x) = \prod_{j=1}^{n} (1 - 4x_j^2)_+,
\]
where
\[
f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0. \end{cases}
\]

For positive integers \( M \) and \( N \), we denote the cube \([-N/2M, N/2M]^n \) by \( NQ/M \). Noting that \( \tilde{T}_R g(x) \) is a periodic function, for large \( N \) we have
\[
\| \tilde{T}_R g \|_{L^p(\mathbb{T}^n)}^p \approx N^{-n} \int_{NQ/2} |\tilde{T}_R g(x)|^p dx.
\]

Since on \( NQ/2 \), there exists a constant \( C > 0 \) such that \( \Psi(x/N) \geq C \), it is easy to see that
\[
\| \tilde{T}_R g \|_{L^p(\mathbb{T}^n)}^p \approx N^{-n} \int_{NQ/2} |\Psi(x/N)\tilde{T}_R g(x)|^p dx.
\]

By Lemma 6 and the assumption of the theorem, we have
\[
\| \tilde{T}_R g \|_{L^p(\mathbb{T}^n)}^p \leq C N^{-n} \int_{\mathbb{R}^n} |\tilde{T}^* (g \Psi^{1/N}(x))|^p dx
\]
\[
+ c N^{-n} \int_{NQ/2} \sup_{0 < \xi \leq R} |J_{N, e}(x)|^p dx
\]
\[
\leq C N^{-n} \| g \Psi^{1/N} \|_{H^p(\mathbb{R}^n)}^p + o(1), \quad \text{as } N \to \infty.
\]
Now it suffices to show that for odd $N$

\[(10) \quad \liminf_{N \to \infty} N^{-n} \|g \Psi^{1/N}\|_{H^p(\mathbb{R}^n)}^p \leq C \|g\|_{H^p(\mathbb{T}^n)}^p.\]

By Lemma 4, we only need to prove that for any $(p, 2, s)$ periodic atom $a(x)$ with support in $B(x_0, \rho) \subset Q$,

\[(11) \quad N^{-n} \|a \Psi^{1/N}\|_{H^p(\mathbb{R}^n)}^p \leq C,
\]

where $C$ is a constant independent of $a(x)$ and $N$.

By the definition, we have

\[N^{-n} \|a \Psi^{1/N}\|_{H^p(\mathbb{R}^n)}^p \geq \int_{\mathbb{R}^n} \sup_{0 < t < \infty} \left| \int_{\mathbb{R}^n} \Psi(x/N) a(x) \Phi_{t}(y - x) \, dx \right|^p \, dy \]

\[= \int_{\mathbb{R}^n} \sup_{0 < t < \infty} \left| \int_{\mathbb{R}^n} \prod_{j=1}^{n} (1 - 4x^2_j/N^2) a(x) \Phi_{t}(y - x) \, dx \right|^p \, dy \]

\[= \int_{\mathbb{R}^n} \sup_{0 < t < \infty} \left| \int_{Q_k} \prod_{j=1}^{n} (1 - 4x^2_j/N^2) a(x) \Phi_{t}(y - x) \, dx \right|^p \, dy.
\]

Now we write $N = 2m + 1$. Then, up to a set of measure 0, the set \( \{ x \in \mathbb{R}^n : |x_j| < m + 1/2, j = 1, 2, \ldots, n \} \) is the union of the disjoint sets \( \{ Q + k : k = (k_1, \ldots, k_n), -m \leq k_j \leq m, j = 1, 2, \ldots, n \} = \{ Q_k \} \), where the $k_j$'s are integers. Now the last integral above is bounded by

\[C \sum_{-m \leq k_j \leq m} \int_{\mathbb{R}^n} \sup_{0 < t < \infty} \left| \int_{Q_k} \prod_{j=1}^{n} (1 - 4x^2_j/N^2) a(x) \Phi_{t}(y - x) \, dx \right|^p \, dy.
\]

Noting that $a(x)$ is a periodic function, we easily see that $\chi_{Q_k}(x)a(x)$ is an atom with support in $Q_k$, where $\chi_{Q_k}$ is the characteristic function of $Q_k$. Also since on $Q_k$, $\prod_{j=1}^{n} (1 - 4x^2_j/N^2)$ is a polynomial of degree $2n$ which is bounded by 1, clearly

\[a(x) = \prod_{j=1}^{n} (1 - 4x^2_j/N^2) \chi_{Q_k}(x)a(x)
\]

is a $(p, 2, [n(1/p - 1)])$ atom on $\mathbb{R}^n$. So the above integral $I_m$ is bounded by

\[C \sum_{-m \leq k_j \leq m} \|a\|_{H^p(\mathbb{R}^n)} \leq C.
\]

Theorem 2 is proved.

Following the proof on page 133 in [LL], we now easily obtain an improvement of Theorem 2 in [LL].

**Theorem 3.** Let $0 < p \leq 1$, and let $1 \leq d < n$ be an integer. Suppose that $\lambda$ is a bounded and continuous function on $\mathbb{R}^n$. If $\lambda$ is a maximal multiplier on $H^p(\mathbb{R}^n)$ $(H^p(\mathbb{T}^n))$, then the restriction of $\lambda$ to $\mathbb{R}^d$ is a maximal multiplier on $H^p(\mathbb{R}^d)$ $(H^p(\mathbb{T}^d))$.

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