TRANSFERENCE OF MAXIMAL MULTIPLIERS ON HARDY SPACES

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ABSTRACT. Based on the atomic decomposition of the Hardy space, we give a simple proof for a theorem of Liu and Lu (Studia Math. 105 (1993), 121-134), which discusses the relation between the maximal operators on $\mathbb{R}^n$ and on $\mathbb{T}^n$. More significantly, our proof shows that condition (1) in Liu and Lu's Theorem 1 is superfluous.

1. INTRODUCTION

Let $H^p(\mathbb{R}^n)$, $0 < p < \infty$, be the Hardy spaces defined by [FS]

$$H^p(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n), \| \Phi^+ f \|_{L^p(\mathbb{R}^n)} < \infty \},$$

where $\Phi^+ f(x) = \sup_{t>0} |\Phi_t * f(x)|$, $\Phi_t(x) = t^{-n}\Phi(x/t)$, and $\Phi \in \mathcal{S}'(\mathbb{R}^n)$ is a radial function satisfying $\int \Phi = 1$. The corresponding periodic Hardy spaces are $H^p(\mathbb{T}^n) = \{ f \in \mathcal{S}'(\mathbb{T}^n), \| \Phi^+ f \|_{L^p(\mathbb{T}^n)} < \infty \}$, where $\Phi^+ f(x) = \sup_{t>0} |\Phi_t * f(x)|$, $\Phi_t(x) = \sum_{k \in \Lambda} \Phi(tk)e^{2\pi ik \cdot x} = Ct^{-n} \sum_{k \in \Lambda} \Phi((x+k)/t)$ and $\Lambda$ is the unit lattice which is the additive group of points in $\mathbb{R}^n$ having integral coordinates.

Let $\lambda$ be a bounded continuous function on $\mathbb{R}^n$. For each $\varepsilon > 0$, define

$$(T_\varepsilon f)(u) = \lambda(\varepsilon u) \hat{f}(u), \quad f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n),$$

and

$$\tilde{T}_\varepsilon f(x) = \sum_{k \in \Lambda} \lambda(ek) a_k(f) e^{2\pi ik \cdot x}, \quad f \in L^2(\mathbb{T}^n) \cap H^p(\mathbb{T}^n).$$

We say that $\lambda$ is a maximal multiplier on $H^p(\mathbb{R}^n)$ if $T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|$ can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Similarly, $\lambda$ is called a maximal multiplier on $H^p(\mathbb{T}^n)$ if $\tilde{T}^* f(x) = \sup_{\varepsilon > 0} |\tilde{T}_\varepsilon f(x)|$ can be extended to a bounded operator from $H^p(\mathbb{T}^n)$ to $L^p(\mathbb{T}^n)$.

The relation between the maximal multipliers $T^*$ and $\tilde{T}^*$ was first studied by Kenig and Tomas when $p > 1$. Their result can be extended to the Lorentz
space \( L(p, q) \), \( p > 1 \) (see [F]). Recently, Liu and Lu [LL] studied the case of \( 0 < p \leq 1 \). Their main result is the following theorem.

**Theorem 1** [LL]. Let \( 0 < p \leq 1 \), and let \( \lambda \) be a bounded and continuous function on \( \mathbb{R}^n \).
(i) Suppose that \( \lambda \) is a maximal multiplier on \( H^p(\mathbb{R}^n) \) such that
\[
\lim_{|x| \to \infty} \lambda(x) = \alpha
\]
exists. Then \( \lambda \) is a maximal multiplier on \( H^p(\mathbb{T}^n) \).
(ii) If \( \lambda \) is a maximal multiplier on \( H^p(\mathbb{T}^n) \), then \( \lambda \) is a maximal multiplier on \( H^p(\mathbb{R}^n) \).

Since the above condition (1) plays an important role in their proof, Liu and Lu asked (see p. 133 in [LL]) if condition (1) can be weakened any further.

In this note, we will show that condition (1) in Theorem 1 is superfluous.

Based on the atomic decomposition of the Hardy space, our proof is much shorter and more direct than those in [LL]. The following is our main result.

**Theorem 2.** Let \( \lambda \) be a continuous and bounded function on \( \mathbb{R}^n \), \( 0 < p \leq 1 \). If
\[
\| T^* f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{H^p(\mathbb{R}^n)} \quad \text{for all } f \in H^p(\mathbb{R}^n),
\]
then
\[
\| \mathcal{T}^* \hat{f} \|_{L^p(\mathbb{T}^n)} \leq C \| \hat{f} \|_{H^p(\mathbb{T}^n)} \quad \text{for all } \hat{f} \in H^p(\mathbb{T}^n).
\]

For the sake of simplicity, \( C \) always denotes a positive constant which may vary at each of its occurrences.

To prove Theorem 2, we need to use the atomic characterization of the Hardy space. A regular \((p, 2, s)\) atom is a function \( a(x) \) supported in some ball \( B(x_0, \rho) \) satisfying
(i) \( \| a \|_2 \leq \rho^{-n/p+n/2} \);
(ii) \( \int_{\mathbb{R}^n} a(x) P(x) \, dx = 0 \)
for all polynomials \( P(x) \) of degree less than or equal to \( s \).

The space \( H^p_{2, s}(\mathbb{R}^n) \), \( 0 < p \leq 1 \), is the space of all distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) having the form
\[
f = \sum c_k a_k
\]
and satisfying
\[
\sum |c_k|^p < \infty,
\]
where each \( a_k \) is a \((p, 2, s)\) atom. The "norm" \( \| f \|_{H^p_{2, s}(\mathbb{R}^n)} \) is the infimum of all expressions \( (\sum |c_k|^p)^{1/p} \) for which we have a representation (2) of \( f \). A well-known fact (see [FoS]) is that \( \| f \|_{H^p_{2, s}(\mathbb{R}^n)} \cong \| \hat{f} \|_{H^p(\mathbb{T}^n)} \), and in particular, \( \| a \|_{H^p(\mathbb{R}^n)} \leq C \), with a constant \( C \) independent of the \((p, 2, s)\) atom \( a(x) \) if \( s \geq \lceil n(1/p - 1) \rceil \).

We also have a similar decomposition theorem for any function \( g \in H^p(\mathbb{T}^n) \). In particular, suppose \( g \in H^p(\mathbb{T}^n) \cap \mathcal{S}'(\mathbb{T}^n) \) and its Fourier coefficient
\[
a_0(g) = \int_{\mathbb{Q}} g(x) \, dx = 0,
\]
where \( Q = \{ x \in \mathbb{R}^n : -1/2 \leq x_j < 1/2, j = 1, 2, \ldots, n \} \) is the fundamental cube on which
\[
\int_{\mathbb{T}^n} g(x) \, dx = \int_Q g(x) \, dx
\]
for all functions \( g \) on \( \mathbb{T}^n \). Then we have the following lemma.

**Lemma 4.** Suppose \( g \in H^p(\mathbb{T}^n) \cap \mathcal{S}(\mathbb{T}^n) \) with \( a_0(g) = 0 \). If we restrict \( x \) to \( Q \), then for any fixed positive integer \( s \)
\[
g(x) = \sum c_k a_k(x),
\]
where each \( a_k(x) \) is a \((p, 2, s)\) atom satisfying \( a_k(x + n) = a_k(x) \) for \( n \in \Lambda \) and \( \|g\|_{H^p(\mathbb{T}^n)} \leq \sum |c_k|^p \).

**Proof.** Choose a radial function \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{supp}(\varphi) \subset B(0, 1) \). In addition, we can choose such a \( \varphi \) such that \( \int x^J \varphi(x) \, dx = 0 \) for all multi-indices \( J, |J| \leq s \), and \( \int_0^\infty \varphi(tx)^2 t^{-1} \, dt = 1 \) for all \( x \neq 0 \). We let \( \tilde{\varphi}_t(x) = \sum_{k \in \Lambda \setminus \{0\}} \widetilde{\varphi}(tk)e^{2\pi ik \cdot x} = C \sum_{k \in \Lambda} t^{-n} \varphi((x+k)/t) \). Then by checking the Fourier coefficients, we easily obtain the following Calderón reproducing formula:
\[
(5) \quad g(x) = \int_0^\infty (\tilde{\varphi}_t * \varphi_t * g)(x) t^{-1} \, dt = \int_0^1 + \int_1^\infty.
\]
Now by a standard argument [FoS] (or see [BF] for the proof on any compact Lie group), one can easily obtain that
\[
g(x) = \sum c_k a_k(x),
\]
where each \( a_k(x) \) is a \((p, 2, s)\) atom satisfying \( a_k(x + n) = a_k(x) \) for \( n \in \Lambda \) and \( \sum |c_k|^p \equiv \|S_\varphi(g)\|_{L^p(\mathbb{T}^n)}^p \). Here \( S_\varphi(g) \) is defined by
\[
S_\varphi(g)(x) = \int_{|x-y| < 1} |(g * \varphi_t)(y)|^2 t^{-n-1} \, dy \, dt.
\]
So to prove the lemma it suffices to show that \( \|S_\varphi(g)\|_{L^p(\mathbb{T}^n)} \equiv \|g\|_{H^p(\mathbb{T}^n)} \) for all \( g \in H^p(\mathbb{T}^n) \cap \mathcal{S}(\mathbb{T}^n) \). But the proof for \( \|S_\varphi(g)\|_{L^p(\mathbb{T}^n)} \equiv \|g\|_{H^p(\mathbb{T}^n)} \) is, mutatis mutandis, the same as for \( \mathbb{R}^n \) (see [FoS]) without using any new techniques or ideas.

The following lemma is Lemma 3.1 in [F]. For completeness, we state its proof.

**Lemma 6.** Suppose that \( \Psi(x) \) is a continuous function with compact support. Let \( \lambda(x) \) be a bounded and continuous function on \( \mathbb{R}^n \), and let \( T_\varepsilon \) and \( \widetilde{T}_\varepsilon \) be the families of operators on \( \mathbb{R}^n \) and \( \mathbb{T}^n \), respectively, associated to the function \( \lambda \). Take \( \Psi^{1/N}(\xi) = \Psi(\xi/N) \). If \( \Psi \) satisfies \( \Psi(0) = 1 \) and \( \Psi \in L^1(\mathbb{R}^n) \), then for any \( g \in \mathcal{S}(\mathbb{T}^n) \) and any positive integer \( N \),
\[
(7) \quad \Psi(y/N)(\widetilde{T}_\varepsilon g)(y) = T_\varepsilon(g \Psi^{1/N})(y) + J_{N, \varepsilon}(y)
\]
for all \( y \in \mathbb{R}^n \), where \( J_{N, \varepsilon}(y) \) tends to zero uniformly for \( y \in \mathbb{R}^n \) and \( 0 \leq \varepsilon \leq R \) \( (R > 0 \) is any fixed number), as \( N \to \infty \).

**Proof.** Since \( g(x) = \sum a_k(g)e^{2\pi ik \cdot x} \) with the Fourier coefficients \( \{a_k(g)\} \) rapidly decreasing as \( |k| \to \infty \), it suffices to prove the lemma when \( g(x) =
\( e_k(x) = e^{2\pi i k \cdot x} \). In this case

\[
|J_{N, \epsilon}(y)| = |\Psi^{1/N}(y)\tilde{T}_\epsilon e_k(y) - T_\epsilon(\Psi^{1/N} e_k)(y)|
\]

\[
= |e_k(y) \int_{\mathbb{R}^n} N^n \tilde{\psi}(N \xi) e^{2\pi i y \cdot \xi} \{\lambda(\epsilon k) - \lambda(\epsilon k + e \xi)\} d\xi|
\]

\[
\leq \int_{\mathbb{R}^n} |\tilde{\psi}(\xi)| |\lambda(\epsilon k) - \lambda(\epsilon k + N^{-1} e \xi)| d\xi.
\]

Since \( \tilde{\psi} \) is integrable and since \( \lambda \) is bounded and continuous, the last quantity converges to zero as \( N \to \infty \). The lemma is proved.

2. Proof of the main theorem

By a note on page 128 in [LL] we only need to show that for any \( g \in \mathcal{S}(\mathbb{T}^n) \cap H^p(\mathbb{T}^n) \) with \( a_0(g) = 0 \),

\[
\|\tilde{T}^* g\|_{L^p(\mathbb{T}^n)} \leq C\|g\|_{H^p(\mathbb{T}^n)}.
\]

For any \( R > 0 \) fixed, we define \( \tilde{T}_R g(x) = \sup_{0 < \epsilon \leq R} |\tilde{T}_\epsilon g(x)| \). Since as \( R \to \infty \) \( \tilde{T}_R g(x) \) increases pointwise to \( \tilde{T}^* g(x) \), by monotonic convergence theorem, to prove (8) we only need to prove that

\[
\|\tilde{T}_R g\|_{L^p(\mathbb{T}^n)} \leq C\|g\|_{H^p(\mathbb{T}^n)}
\]

with a constant \( C \) independent of \( R \) and \( g(x) \).

By Lemma 4, \( g(x) = \sum c_k a_k(x) \), where each \( a_k \) is a \([n(1/p - 1)] + 2n\) atom and \( \sum |c_k|^p \approx \|g\|_{H^p(\mathbb{T}^n)}^p \).

We let

\[
\Psi(x) = \prod_{j=1}^n \left(1 - 4x_j^2\right)_+,
\]

where

\[
f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0. \end{cases}
\]

For positive integers \( M \) and \( N \), we denote the cube \([-N/2M, N/2M]^n\) by \( NQ/M \). Noting that \( \tilde{T}_R g(x) \) is a periodic function, for large \( N \) we have

\[
\|\tilde{T}_R g\|_{L^p(\mathbb{T}^n)}^p \approx N^{-n} \int_{NQ/2} |\tilde{T}_R g(x)|^p dx.
\]

Since on \( NQ/2 \), there exists a constant \( C > 0 \) such that \( \Psi(x/N) \geq C \), it is easy to see that

\[
\|\tilde{T}_R g\|_{L^p(\mathbb{T}^n)}^p \approx N^{-n} \int_{NQ/2} |\Psi(x/N)\tilde{T}_R g(x)|^p dx.
\]

By Lemma 6 and the assumption of the theorem, we have

\[
\|\tilde{T}_R g\|_{L^p(\mathbb{T}^n)}^p \leq CN^{-n} \int_{\mathbb{R}^n} |T^*(g \Psi^{1/N})(x)|^p dx
\]

\[
+ cN^{-n} \int_{NQ/2} \sup_{0 < \epsilon \leq R} |J_{N, \epsilon}(x)|^p dx
\]

\[
\leq CN^{-n} \|g \Psi^{1/N}\|_{H^p(\mathbb{R}^n)}^p + o(1), \quad \text{as } N \to \infty.
\]
Now it suffices to show that for odd $N$

\[(10) \quad \liminf_{N \to \infty} N^{-n} \|g \Psi^{1/N} \|^p_{H^p(R^n)} \leq C \|g \|^p_{H^p(T^n)}.\]

By Lemma 4, we only need to prove that for any $(p, 2, s)$ periodic atom $a(x)$ with support in $B(x_0, \rho) \subset Q$, \n
\[(11) \quad N^{-n} \|a \Psi^{1/N} \|^p_{H^p(R^n)} \leq C,\]

where $C$ is a constant independent of $a(x)$ and $N$.

By the definition, we have \n
\[N^{-n} \|a \Psi^{1/N} \|^p_{H^p(R^n)} \approx N^{-n} \int_{R^n} \sup_{0 < t < \infty} \left| \int_{R^n} \Psi(x/N) a(x) \Phi_t(y - x) \, dx \right|^p \, dy\]
\[= N^{-n} \int_{R^n} \sup_{0 < t < \infty} \left| \int_{R^n} \prod_{j=1}^n \left(1 - 4x_j^2/N^2\right) a(x) \Phi_t(y - x) \, dx \right|^p \, dy\]
\[= N^{-n} \int_{R^n} \sup_{0 < t < \infty} \left| \int_{|x_j| < N/2} \left\{ \prod_{j=1}^n \left(1 - 4x_j^2/N^2\right) a(x) \right\} \Phi_t(y - x) \, dx \right|^p \, dy.\]

Now we write $N = 2m + 1$. Then, up to a set of measure 0, the set \{ $x \in \mathbb{R}^n : |x_j| < m + 1/2$, $j = 1, 2, \ldots, n$ \} is the union of the disjoint sets \{ $Q + k : k = (k_1, \ldots, k_n)$, $-m \leq k_j \leq m$, $j = 1, 2, \ldots, n$ \} = \{$Q_k$\}, where the $k_j$'s are integers. Now the last integral above is bounded by \n
\[I_m = Cm^{-n} \sum_{-m \leq k_j \leq m} \int_{R^n} \sup_{0 < t < \infty} \left| \int_{Q_k} \left\{ \prod_{j=1}^n \left(1 - 4x_j^2/N^2\right) a(x) \right\} \Phi_t(y - x) \, dx \right|^p \, dy.\]

Noting that $a(x)$ is a periodic function, we easily see that $\chi_{Q_k}(x)a(x)$ is an atom with support in $Q_k$, where $\chi_{Q_k}$ is the characteristic function of $Q_k$. Also since on $Q_k$, $\prod_{j=1}^n \left(1 - 4x_j^2/N^2\right)$ is a polynomial of degree $2n$ which is bounded by 1, clearly \n
\[\alpha(x) = \prod_{j=1}^n \left(1 - 4x_j^2/N^2\right) \chi_{Q_k}(x)a(x)\]

is a $(p, 2, [n(1/p - 1)])$ atom on $\mathbb{R}^n$. So the above integral $I_m$ is bounded by \n
\[Cm^{-n} \sum_{-m \leq k_j \leq m} \|\alpha\|^p_{H^p(R^n)} \leq C.\]

Theorem 2 is proved.

Following the proof on page 133 in [LL], we now easily obtain an improvement of Theorem 2 in [LL].

**Theorem 3.** Let $0 < p \leq 1$, and let $1 \leq d < n$ be an integer. Suppose that $\lambda$ is a bounded and continuous function on $\mathbb{R}^n$. If $\lambda$ is a maximal multiplier on $H^p(\mathbb{R}^n)$ ($H^p(T^n)$), then the restriction of $\lambda$ to $\mathbb{R}^d$ is a maximal multiplier on $H^p(\mathbb{R}^d)$ ($H^p(T^d)$).

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References

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