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PURELY INFINITE SIMPLE C^*-CROSSED PRODUCTS

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ABSTRACT. Let α be an outer action by a finite group G on a simple C^*-algebra A with $RR(A) = 0$. If A is purely infinite, then the C^*-crossed product $A \times_\alpha G$ is purely infinite. The converse is also true if G is a finite abelian group.

1. INTRODUCTION

A C^*-algebra A is said to be *infinite* if A contains an infinite projection p, that is, p is Murray-von Neumann equivalent to its subprojection, and *purely infinite* if every hereditary C^*-subalgebra B of A is infinite. For a simple C^*-algebra A, A is purely infinite if and only if A has a purely infinite hereditary C^*-subalgebra B. In fact, for any hereditary C^*-subalgebra C in A, there exists a unitary u in the multiplier algebra $M(A)$ of A such that $uBu^* \cap C \neq 0$ [8, Lemma 3.4]. Every projection in a purely infinite C^*-algebra is obviously infinite, but it is not known whether there exists an infinite simple C^*-algebra containing a projection which is not infinite. In [11] Zhang proved that a purely infinite simple C^*-algebra A has the following property (FS): the set of all selfadjoint elements with finite spectra is dense in the set of all selfadjoint elements A_{sa} in A, equivalently the set of all invertible selfadjoint elements is dense in A_{sa} ($RR(A) = 0$) [1], which means that purely infinite simple C^*-algebras have many projections in some sense. Cuntz algebras $\mathcal{O}_n$ $(n \geq 2)$, Cuntz-Krieger algebras $\mathcal{O}_A$ (A is an irreducible matrix) [2] and the Calkin algebra $B(H)/\mathcal{K}$ ($\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space H) are examples of purely infinite simple C^*-algebras.

Kishimoto [5] showed that the reduced C^*-crossed product $A \times_\alpha G$ of a simple C^*-algebra by an outer action α of a discrete group G is simple. If A is purely infinite simple, then $A \times_\alpha G$ is obviously infinite simple since it contains A as a C^*-subalgebra. We show that it is actually purely infinite if G is finite. It will also be shown that pure infiniteness of $A \times_\alpha G$ implies that

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of A if A is a simple C^*-algebra of real rank zero ($RR(A) = 0$) and G is a finite abelian group.

2. PURELY INFINITE C^*-CROSSED PRODUCTS

It is known that a simple C^*-algebra A is purely infinite if and only if $RR(A) = 0$ and every projection of A is infinite [13, Theorem 1.2]. In [9] Rørdam proved that if A is a unital C^*-algebra, then $M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})$ is simple if and only if either $A \cong M_n$ or A is purely infinite simple.

Theorem 1. *Let (A, G, α) be a C^*-dynamical system with a finite group G and a unital purely infinite simple C^*-algebra A such that $A \times_\alpha G$ is simple. Then $A \times_\alpha G$ is purely infinite if and only if $RR(A \times_\alpha G) = 0$.*

Proof. Since the purely infinite simple C^*-algebra $A \times_\alpha G$ has real rank zero, that is, $RR(A \times_\alpha G) = 0$, we only need to prove the converse. It suffices to show that $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\alpha \otimes \mathrm{id}} G$ is simple since

$$M((A \times_\alpha G) \otimes \mathscr{K})/((A \times_\alpha G) \otimes \mathscr{K}) \cong (M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\alpha \otimes \mathrm{id}} G.$$

This algebra is isomorphic to $M((A \otimes \mathscr{K}) \times_{\alpha \otimes \mathrm{id}} G)/((A \otimes \mathscr{K}) \times_{\alpha \otimes \mathrm{id}} G)$, which is prime by [12, Theorem 6.2]. For ease of notation write $\tilde{\alpha}$ for $\alpha \otimes \mathrm{id}$. Since $M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})$ is simple, the subgroup $N = \{t \in G : \tilde{\alpha}_t(x) = u_t x u_t^*$ for some unitary $u_t \in M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})\}$ is normal in G [7, p.158]. The primeness of $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} G$ means that $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} N$ is G-prime [7, Proposition 2.5] for the action β of G on $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} N$ defined by

$$\beta_t(\sum_s x_s \lambda_s) = \sum_s \tilde{\alpha}_t(x_s) \lambda_{tst^{-1}}$$

for $\sum_s x_s \lambda_s \in (M(A \otimes \mathscr{K})/(A \otimes K)) \times_{\tilde{\alpha}} N$ and $t \in G$ [7, p.164], where $x_s \in M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})$ and λ is the left regular representation of G. Let C be the commutant of $M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})$ in $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} N$. Then $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} N$ is G-prime if and only if C is G-simple [7, Proposition 2.9]. Since $A \otimes \mathscr{K}$ is purely infinite simple, $M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})$ is simple. In this case the G-invariant ideals of $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} N$ correspond exactly to the G-invariant ideals of C and hence we conclude that $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} N$ is G-simple, therefore $(M(A \otimes \mathscr{K})/(A \otimes \mathscr{K})) \times_{\tilde{\alpha}} G$ is simple [7, Theorem 3.2].

Lemma 2. *Let $\{p_i\}_{i=1}^n$ be finitely many projections in a simple C^*-algebra A such that $\|p_i p_j\| < \varepsilon (< \frac{1}{2n})$, $i \neq j$. Then their supremum $\vee p_i$ (in A^{**}) is a projection in A and $\| \sum_i p_i - \vee p_i \| \leq \frac{n^2 \varepsilon}{\sqrt{1 - n\varepsilon}}$.*

Proof. The first assertion is [4, Lemma 2]. Recall that the supremum $\vee p_i$ of $\{p_i\}$ is the projection on the closed subspace $\{p_1 \xi_1 + \cdots + p_n \xi_n | \xi_i \in \mathscr{H}\}^-$ where $(\pi_u, \mathscr{H})$ is the universal representation of A (so that A^{**} is the σ-weak closure of $\pi_u(A)$ in $B(\mathscr{H})$). Let $\xi = p_1 \xi_1 + \cdots + p_n \xi_n$ be a unit vector in $(\vee p_i)\mathscr{H}$. Then from the proof of [4, Lemma 2] it follows that $\|p_i \xi_i\| \leq \frac{1}{\sqrt{1 - n\varepsilon}}$ for each

i. Then we have

$$\begin{aligned}\|(\sum p_i - \vee p_i)\xi\| &= \|\sum_i p_i(\sum_j p_j\xi_j) - \xi\| \\ &= \|\sum_i p_i\xi_i + \sum_{i\neq j} p_i p_j \xi_j - \sum_i p_i\xi_i\| \\ &< \frac{n^2\varepsilon}{\sqrt{1-n\varepsilon}}.\end{aligned}$$

An action α of a group G on a C^*-algebra A is said to be *outer* if each automorphism α_g is outer for each $g \neq 1$, where 1 denotes the unit of G.

Theorem 3. *Let α be an outer action by a finite group G on a simple C^*-algebra A with $RR(A) = 0$. If A is purely infinite, then the C^*-crossed product $A \times_\alpha G$ is purely infinite. Conversely, if $A \times_\alpha G$ is purely infinite and G is abelian, then A is purely infinite.*

Proof. Since G is finite, we can, as in [10], identify the fixed-point algebra A^α with a certain hereditary C^*-subalgebra of the simple C^*-crossed product $A \times_\alpha G$ [10]. Hence it suffices to show that A^α is purely infinite. Let $|G| = n$. For every G-invariant hereditary C^*-subalgebra B of A and every $\varepsilon > 0$, there exists a nonzero projection p in B such that $\|\alpha_s(p)\alpha_t(p)\| < \varepsilon$ for $s \neq t$, $s, t \in G$. Let $\tilde{p} = \vee_{g\in G} \alpha_g(p)$, which is in B by Lemma 2 for $\varepsilon < 1/4n^2$, so in B^α. We prove $\tilde{p}$ is infinite in B^α.

Since A is purely infinite, there is a partial isometry v in A such that $v^*v = p$, $vv^* = e < p$ and $v = ev = vp$. Put $w = \sum_{g\in G} \alpha_g(v) \in A^\alpha$. Then

$$\begin{aligned}w^*w &= (\sum_{g\in G} \alpha_g(v^*))(\sum_{s\in G} \alpha_s(v)) \\ &= \sum_{g,s\in G} \alpha_g(v^*)\alpha_s(v) \\ &= \sum_{g\in G} \alpha_g(p) + \sum_{g\neq s} \alpha_g(v^*)\alpha_s(v).\end{aligned}$$

Since $\|\sum_{g\neq s} \alpha_g(v^*)\alpha_s(v)\| = \|\sum_{g\neq s} \alpha_g(v^*)\alpha_g(e)\alpha_s(e)\alpha_s(v)\| < n^2\varepsilon$, it follows from Lemma 2 that

$$\|w^*w - \tilde{p}\| \leq \|w^*w - \sum_{g\in G}\alpha_g(p)\| + \|\sum_{g\in G}\alpha_g(p) - \tilde{p}\| < n^2\varepsilon + \frac{n^2\varepsilon}{\sqrt{1-n\varepsilon}} < 1.$$

Hence w^*w is invertible in the hereditary C^*-subalgebra $(A^\alpha)_{\tilde{p}}$ of A^α generated by $\tilde{p}$ with the inverse w_1. Then $u = w(w_1)^{1/2} \in (A^\alpha)_{\tilde{p}}$ is a partial isometry such that $u^*u = \tilde{p}$ and $uu^* < \tilde{p}$, and $\tilde{p}$ is infinite in A^α.

If G is abelian, then it follows from the duality theorem for crossed products that $(A \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ is isomorphic to $A \otimes M_n$ which is of real rank zero. By Theorem 1 we conclude that A is purely infinite if $A \times_\alpha G$ is purely infinite.

Example 4. In [3] it was proved that every countable discrete group has a faithful representation as a subgroup of outer automorphisms of the Cuntz algebra $\mathcal{O}_\infty$. If a finite group G acts on $\mathcal{O}_n = C^*(S_1, \cdots, S_n)$ by permutations on

$\{S_1, \cdots, S_n\}$ $(2 \leq n \leq \infty)$, then the action is outer [3, 6] and the simple C^*-crossed product is purely infinite by Theorem 2.

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