PURELY INFINITE SIMPLE $C^*$-CROSSED PRODUCTS

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ABSTRACT. Let $\alpha$ be an outer action by a finite group $G$ on a simple $C^*$-algebra $A$ with $RR(A) = 0$. If $A$ is purely infinite, then the $C^*$-crossed product $A \times_\alpha G$ is purely infinite. The converse is also true if $G$ is a finite abelian group.

1. INTRODUCTION

A $C^*$-algebra $A$ is said to be infinite if $A$ contains an infinite projection $p$, that is, $p$ is Murray-von Neumann equivalent to its subprojection, and purely infinite if every hereditary $C^*$-subalgebra $B$ of $A$ is infinite. For a simple $C^*$-algebra $A$, $A$ is purely infinite if and only if $A$ has a purely infinite hereditary $C^*$-subalgebra $B$. In fact, for any hereditary $C^*$-subalgebra $B$ in $A$, there exists a unitary $u$ in the multiplier algebra $M(A)$ of $A$ such that $uBu^* \cap C \neq 0$ [8, Lemma 3.4]. Every projection in a purely infinite $C^*$-algebra is obviously infinite, but it is not known whether there exists an infinite simple $C^*$-algebra containing a projection which is not infinite. In [11] Zhang proved that a purely infinite simple $C^*$-algebra $A$ has the following property (FS): the set of all selfadjoint elements with finite spectra is dense in the set of all selfadjoint elements $A_{sa}$ in $A$, equivalently the set of all invertible selfadjoint elements is dense in $A_{sa} (RR(A) = 0)$ [1], which means that purely infinite simple $C^*$-algebras have many projections in some sense. Cuntz algebras $\mathcal{O}_n \ (n \geq 2)$, Cuntz-Krieger algebras $\mathcal{O}_A \ (A$ is an irreducible matrix) [2] and the Calkin algebra $B(H)/\mathcal{K} \ (\mathcal{K}$ is the $C^*$-algebra of compact operators on a separable infinite-dimensional Hilbert space $H)$ are examples of purely infinite simple $C^*$-algebras.

Kishimoto [5] showed that the reduced $C^*$-crossed product $A \times_\alpha G$ of a simple $C^*$-algebra by an outer action $\alpha$ of a discrete group $G$ is simple. If $A$ is purely infinite simple, then $A \times_\alpha G$ is obviously infinite simple since it contains $A$ as a $C^*$-subalgebra. We show that it is actually purely infinite if $G$ is finite. It will also be shown that pure infiniteness of $A \times_\alpha G$ implies that...
of $A$ if $A$ is a simple $C^*$-algebra of real rank zero ($RR(A) = 0$) and $G$ is a finite abelian group.

2. Purely infinite $C^*$-crossed products

It is known that a simple $C^*$-algebra $A$ is purely infinite if and only if $RR(A) = 0$ and every projection of $A$ is infinite [13, Theorem 1.2]. In [9] Rørdam proved that if $A$ is a unital $C^*$-algebra, then $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is simple if and only if either $A \cong M_n$ or $A$ is purely infinite simple.

**Theorem 1.** Let $(A, G)$ be a $C^*$-dynamical system with a finite group $G$ and a unital purely infinite simple $C^*$-algebra $A$ such that $A \rtimes G$ is simple. Then $A \rtimes G$ is purely infinite if and only if $RR(A \rtimes G) = 0$.

**Proof.** Since the purely infinite simple $C^*$-algebra $A \rtimes G$ has real rank zero, that is, $RR(A \rtimes G) = 0$, we only need to prove the converse. It suffices to show that $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\alpha \otimes \text{id}} G$ is simple since

$$M((A \rtimes G) \otimes \mathcal{K})/((A \rtimes G) \otimes \mathcal{K}) \cong (M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\alpha \otimes \text{id}} G.$$  

This algebra is isomorphic to $M((A \otimes \mathcal{K}) \times_{\alpha \otimes \text{id}} G)/((A \otimes \mathcal{K}) \times_{\alpha \otimes \text{id}} G)$, which is prime by [12, Theorem 6.2]. For ease of notation write $\tilde{\alpha}$ for $\alpha \otimes \text{id}$. Since $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is simple, the subgroup $N = \{t \in G : \tilde{a}_t(x) = u_t x u_t^* \text{ for some unitary } u_t \in M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})\}$ is normal in $G$ [7, p. 158]. The primeness of $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} G$ means that $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} N$ is $G$-prime [7, Proposition 2.5] for the action $\beta$ of $G$ on $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} N$ defined by

$$\beta_t^s(\sum_s x_s \lambda_s) = \sum_s \tilde{a}_t(x_s) \lambda_{ts^{-1}}$$

for $\sum_s x_s \lambda_s \in (M(A \otimes \mathcal{K})/(A \otimes K)) \times_{\tilde{\alpha}} N$ and $t \in G$ [7, p.164], where $x_s \in M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ and $\lambda$ is the left regular representation of $G$. Let $C$ be the commutant of $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ in $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} N$. Then $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} N$ is $G$-prime if and only if $C$ is $G$-simple [7, Proposition 2.9]. Since $A \otimes \mathcal{K}$ is purely infinite simple, $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is simple. In this case the $G$-invariant ideals of $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} N$ correspond exactly to the $G$-invariant ideals of $C$ and hence we conclude that $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} N$ is $G$-simple, therefore $(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})) \times_{\tilde{\alpha}} G$ is simple [7, Theorem 3.2].

**Lemma 2.** Let $\{p_i\}_{i=1}^n$ be finitely many projections in a simple $C^*$-algebra $A$ such that $\|p_i p_j\| < \varepsilon( < 1/n_i)$, $i \neq j$. Then their supremum $\vee p_i$ (in $A^{**}$) is a projection in $A$ and $\|\sum_i p_i - \vee p_i\| \leq \frac{n \varepsilon}{\sqrt{1-n \varepsilon}}$.

**Proof.** The first assertion is [4, Lemma 2]. Recall that the supremum $\vee p_i$ of $\{p_i\}$ is the projection on the closed subspace $\{p_1 \xi_1 + \cdots + p_n \xi_n | \xi_i \in \mathcal{K}\}$ where $(\pi_u, \mathcal{K})$ is the universal representation of $A$ (so that $A^{**}$ is the $\sigma$-weak closure of $\pi_u(A)$ in $B(\mathcal{K})$). Let $\xi = p_1 \xi_1 + \cdots + p_n \xi_n$ be a unit vector in $(\vee p_i) \mathcal{K}$. Then from the proof of [4, Lemma 2] it follows that $\|p_i \xi_i\| \leq \frac{1}{\sqrt{1-n \varepsilon}}$ for each
Then we have

\[ \| (\sum_i p_i - \sqrt{p_i}) \xi \| = \| \sum_i p_i (\sum_j p_j \xi_j) - \xi \| \]
\[ = \| \sum_i p_i \xi_i + \sum_{i \neq j} p_i p_j \xi_j - \sum_i p_i \xi_i \| \]
\[ < \frac{n^2 \epsilon}{\sqrt{1 - ne}}. \]

An action \( \alpha \) of a group \( G \) on a \( C^* \)-algebra \( A \) is said to be outer if each automorphism \( \alpha_g \) is outer for each \( g \neq 1 \), where \( 1 \) denotes the unit of \( G \).

**Theorem 3.** Let \( \alpha \) be an outer action by a finite group \( G \) on a simple \( C^* \)-algebra \( A \) with \( RR(A) = 0 \). If \( A \) is purely infinite, then the \( C^* \)-crossed product \( A \times_\alpha G \) is purely infinite. Conversely, if \( A \times_\alpha G \) is purely infinite and \( G \) is abelian, then \( A \) is purely infinite.

**Proof.** Since \( G \) is finite, we can, as in [10], identify the fixed-point algebra \( A^\alpha \) with a certain hereditary \( C^* \)-subalgebra of the simple \( C^* \)-crossed product \( A \times_\alpha G \) [10]. Hence it suffices to show that \( A^\alpha \) is purely infinite. Let \( |G| = n \).

For every \( G \)-invariant hereditary \( C^* \)-subalgebra \( B \) of \( A \) and every \( \epsilon > 0 \), there exists a nonzero projection \( p \) in \( B \) such that \( \| \alpha_s(p) \alpha_t(p) \| < \epsilon \) for \( s \neq t \), \( s, t \in G \). Let \( \tilde{p} = \vee g \in G \alpha_g(p) \), which is in \( B \) by Lemma 2 for \( \epsilon < 1/4n^2 \), so in \( B^\alpha \). We prove \( \tilde{p} \) is infinite in \( B^\alpha \).

Since \( A \) is purely infinite, there is a partial isometry \( v \) in \( A \) such that \( v^* v = p \), \( vv^* = e < p \) and \( v = ev = vp \). Put \( w = \sum_{g \in G} \alpha_g(v) \in A^\alpha \). Then

\[ w^* w = (\sum_{g \in G} \alpha_g(v^*)) (\sum_{s \in G} \alpha_s(v)) \]
\[ = \sum_{g \neq s \in G} \alpha_g(v^*) \alpha_s(v) \]
\[ = \sum_{g \in G} \alpha_g(p) + \sum_{g \neq s} \alpha_g(v^*) \alpha_s(v). \]

Since \( \| \sum_{g \neq s} \alpha_g(v^*) \alpha_s(v) \| < n^2 \epsilon \), it follows from Lemma 2 that

\[ \| w^* w - \tilde{p} \| \leq \| w^* w - \sum_{g \in G} \alpha_g(p) \| + \| \sum_{g \in G} \alpha_g(p) - \tilde{p} \| < n^2 \epsilon + \frac{n^2 \epsilon}{\sqrt{1 - ne}} < 1. \]

Hence \( w^* w \) is invertible in the hereditary \( C^* \)-subalgebra \( (A^\alpha)_{\tilde{p}} \) of \( A^\alpha \) generated by \( \tilde{p} \) with the inverse \( w_1 \). Then \( u = w(w_1)^{1/2} \in (A^\alpha)_{\tilde{p}} \) is a partial isometry such that \( u^* u = \tilde{p} \) and \( uu^* < \tilde{p} \), and \( \tilde{p} \) is infinite in \( A^\alpha \).

If \( G \) is abelian, then it follows from the duality theorem for crossed products that \( (A \times_\alpha G) \times_\alpha \tilde{G} \) is isomorphic to \( A \otimes M_n \) which is of real rank zero. By Theorem 1 we conclude that \( A \) is purely infinite if \( A \times_\alpha G \) is purely infinite.

**Example 4.** In [3] it was proved that every countable discrete group has a faithful representation as a subgroup of outer automorphisms of the Cuntz algebra \( \mathcal{O}_\infty \). If a finite group \( G \) acts on \( \mathcal{O}_n = C^*(S_1, \ldots, S_n) \) by permutations on
If \( \{S_1, \cdots, S_n\} \) \( (2 \leq n \leq \infty) \), then the action is outer [3, 6] and the simple \( C^* \)-crossed product is purely infinite by Theorem 2.

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REFERENCES


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