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# THE BOREL CLASSES OF MAHLER'S $A$, $S$, $T$, AND $U$ NUMBERS

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ABSTRACT. In this article we examine the $A$, $S$, $T$, and $U$ sets of Mahler's classification from a descriptive set theoretic point of view. We calculate the possible locations of these sets in the Borel hierarchy. $A$ turns out to be $\mathbf{\Sigma}_2^0$-complete, while $U$ provides a rare example of a natural $\mathbf{\Sigma}_3^0$-complete set. We produce an upperbound of $\mathbf{\Sigma}_4^0$ for $S$ and show that $T$ is $\mathbf{\Pi}_4^0$ but not $\mathbf{\Sigma}_3^0$. Our main result is based on a deep theorem of Schmidt that allows us to guarantee the existence of the $T$ numbers.

## INTRODUCTION

Mahler [6] divided complex numbers into classes $A$, $S$, $T$, and $U$ according to their properties of approximation by algebraic numbers. Some studies were done on the structural properties of these sets. For example, Kasch and Volkmann [3] verified that the $T$ numbers have Hausdorff dimension zero. Also in harmonic analysis, W. Morgan, C. E. M. Pearce, and A. D. Pollington [7] have shown that the set of $T$ and $U$ numbers supports a measure whose Fourier transform vanishes at infinity. In the present paper we study the $A$, $S$, $T$, and $U$ sets from the point of view of Descriptive Set Theory. Among the few sets whose exact Borel class is known, a large percentage turn out to be $\mathbf{\Pi}_3^0$-complete. For example, the collection of reals that are normal or simply normal to base $n$ [4]; $C^\infty(\mathbb{T})$, the class of infinitely differentiable functions (viewed as a $2\pi$-periodic function on $\mathbb{R}$); and $UC_X$, the class of convergent sequences in a separable Banach space $X$ are $\mathbf{\Pi}_3^0$-complete [2]. Apparently, there are few known natural $\mathbf{\Sigma}_3^0$-complete sets. Of course, the complement of a $\mathbf{\Pi}_3^0$-complete set is $\mathbf{\Sigma}_3^0$-complete. But the complement of a natural set need not be natural! Tom Linton [5] has shown that the family of $H$-sets, a class of thin sets from harmonic analysis, is $\mathbf{\Sigma}_3^0$-complete, and this is the only $\mathbf{\Sigma}_3^0$-complete natural set we know of (whose complement is not also natural). A. Kechris proposed to find out what the Borel classes of the $A$, $S$, $T$, and $U$ sets are. It turns out that $A$ is rather simple, being $\mathbf{\Sigma}_2^0$-complete. On the other hand, $T$ is $\mathbf{\Pi}_3^0$-hard, while $U$ is $\mathbf{\Sigma}_3^0$-complete. Our main results are based on a theorem of W. M.

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Schmidt (see [1, pp. 85–94]). The exact Borel classes of the $S$ and $T$ sets are unknown to us.

## DEFINITIONS AND BACKGROUND

For spaces $X$ and $Y$, $X^Y$ denotes the set of all functions $f$ from $Y$ to $X$, with the usual product topology, $X$ and $Y$ being endowed with their usual topologies ($2 = \{0, 1\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$ being discrete). For sets $U$ and $V$, if $S$ is a function from $X^{n+1} \times Y^{n+1}$ to $U^{n+1} \times V^{n+1}$ and $n \in \mathbb{N}$, then $S|_n$ is the function from $X^{n+1} \times Y^{n+1}$ to $U^n \times V^n$ such that if $S((x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1})) = ((u_1, \dots, u_{n+1}), (v_1, \dots, v_{n+1}))$, then $S|_n((x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1})) = ((u_1, \dots, u_n), (v_1, \dots, v_n))$. $\mathbb{P} = \{x \in \mathbb{R} : x > 1\}$ and $\mathbb{A}$ denotes the class of all nonzero real algebraic numbers in $\mathbb{C}$. We use the standard terminology of Addison to describe the Borel hierarchy. Thus the multiplicative sets of level $n$ are denoted by $\mathbf{\Pi^0_n}$, while the additive class of level $n$ is denoted by $\mathbf{\Sigma^0_n}$. In particular, $\mathbf{\Sigma^0_1} = \text{Open}$, $\mathbf{\Pi^0_1} = \text{Closed}$, $\mathbf{\Sigma^0_2} = F_\sigma$, $\mathbf{\Pi^0_2} = G_\delta$. In addition, the countable union of $\mathbf{\Pi^0_n}$ sets is $\mathbf{\Sigma^0_{n+1}}$; the countable intersection of $\mathbf{\Pi^0_n}$ sets is a $\mathbf{\Sigma^0_{n+1}}$ set; the complement of a $\mathbf{\Pi^0_n}$ set is $\mathbf{\Sigma^0_n}$; the $\mathbf{\Sigma^0_n}$ sets are closed under finite intersection and countable union; while the $\mathbf{\Pi^0_n}$ sets are closed under finite union and countable intersection. If the context demands it, we use $\mathbf{\Pi^0_n}(X)$ to denote the $\mathbf{\Pi^0_n}$ subsets of a space $X$.

Let $\Gamma = \mathbf{\Sigma^0_n}$ or $\mathbf{\Pi^0_n}$. We call a set $C \subseteq X$ (a Polish space) $\Gamma$-hard if for any $B \in \Gamma(2^{\mathbb{N}})$, there is a continuous function $f$ from $2^{\mathbb{N}}$ to $X$, such that $B = f^{-1}(C)$. If, moreover, $C \in \Gamma(X)$, we call $C$ $\Gamma$-complete. It is well known (see [2]) that a $\mathbf{\Pi^0_n}$-complete set in an uncountable Polish space is $\mathbf{\Pi^0_n}$ but not $\mathbf{\Sigma^0_n}$, and if $A$ is $\mathbf{\Pi^0_n}$-hard, then $A$ is not $\mathbf{\Sigma^0_n}$. As well, in uncountable Polish spaces every $\mathbf{\Pi^0_n}$ set and every $\mathbf{\Sigma^0_n}$ set is $\mathbf{\Pi^0_{n+1}}$ and $\mathbf{\Sigma^0_{n+1}}$, so the Borel hierarchy is increasing in $n$.

For a given set $C \subseteq X$, in order to find the exact Borel class of $C$, one must first calculate an upperbound for $C$, by showing, for example, that $C$ is $\mathbf{\Pi^0_n}$ and then prove a lowerbound for $C$'s Borel class, for example, by showing that $C$ is $\mathbf{\Pi^0_n}$-hard. Usually, finding the upperbound is fairly easy. However, it can be difficult to prove the hardness of $C$. Since the Borel classes $\mathbf{\Pi^0_n}$ and $\mathbf{\Sigma^0_n}$ are closed under continuous preimages, if $B$ is $\Gamma$-hard ($\Gamma$-complete) and $B = f^{-1}(C)$, where $f$ is a continuous function, then $C$ is $\Gamma$-hard ($\Gamma$-complete, if also $C \in \Gamma$). This remark is the basis of a common method for showing that a given set $B$ is $\Gamma$-hard: Choose an already known $\Gamma$-hard set $B$ and show that there is a continuous function $f$ such that $B = f^{-1}(C)$.

Now we define the $A$, $S$, $T$, and $U$ sets, from Mahler's classification. For convenience we use Koksma's notation, which is equivalent to that of Mahler. Given algebraic $\alpha \in \mathbb{C}$, let $p(x) \in \mathbb{Z}[x]$ be its minimal polynomial. Fix $d, h \in \mathbb{N}$. Let $X_{d,h}$ be the finite collection of polynomials with degree $\leq d$ whose largest coefficient has absolute value $\leq h$. Let the height of a polynomial, $\operatorname{ht}(p)$, be the maximum of the absolute values of the coefficients. Let $A_{d,h}$ be the finite collection of algebraic numbers $\alpha$ such that for some $p \in X_{d,h}$, $p(\alpha)$ is zero (recall that $0 \notin \mathbb{N}$). Thus, $A_{d,h}$ is the finite collection of algebraic (complex) numbers whose minimal polynomial has degree $\leq d$ and $\operatorname{ht} \leq h$. Let $\xi$ be any complex number and let $\alpha$ belong to $A_{d,h}$ such that $|\xi - \alpha|$ takes the smallest positive value; define $\omega_d^*(\xi, h)$ by

$$|\xi - \alpha| = \frac{1}{h^{d\omega_d^*(\xi, h)+1}}.$$

Set

$$\omega_d^*(\xi) = \limsup_{h\to\infty} \omega_d^*(\xi, h) \quad \text{and} \quad \omega^*(\xi) = \limsup_{d\to\infty} \omega_d^*(\xi).$$

So the values of $\omega_d^*(\xi)$ and $\omega^*(\xi)$ measure how fast $\xi$ is approximated by algebraic numbers. We define, according to the values of $\omega_d^*(\xi)$ and $\omega^*(\xi)$, the $A$, $S$, $T$, and $U$ sets as follows:

$$\begin{aligned}
A &= \{\xi \in \mathbb{C} : \omega^*(\xi) = 0\}, \\
S &= \{\xi \in \mathbb{C} : 0 < \omega^*(\xi) < \infty\}, \\
T &= \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty \text{ and } \forall d \in \mathbb{N}\ (\omega_d^*(\xi) < \infty)\}, \\
U &= \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty \text{ and } \exists d \in \mathbb{N}\ (\omega_d^*(\xi) = \infty)\}.
\end{aligned}$$

Thus, the $A$ numbers are slowly approximated by algebraic numbers. The $S$ numbers are approximated a bit more quickly than $A$ numbers. On the other hand, the $T$ numbers and the $U$ numbers are very rapidly approximated, i.e., the value of $\omega^*(\xi)$ is infinite. In particular, the approximation of the $U$ numbers is so quick that for some $d \in \mathbb{N}$, $\omega_d^*(\xi)$ diverges. For these reasons, we claim that the set of complex numbers is naturally partitioned by the $A$, $S$, $T$, and $U$ numbers.

## Results

**Lemma 1.** $\xi \in A \Leftrightarrow \xi$ *is an algebraic number.*

*Proof.* See [1, pp. 85–94].

**Proposition 2.** (i) *The $A$ numbers are $\mathbf{\Sigma}_2^0$-complete, and the $U$ numbers are $\mathbf{\Sigma}_3^0$.*

(ii) *The $S$ numbers are $\mathbf{\Sigma}_4^0$, while the collection of $T$ numbers are $\mathbf{\Pi}_4^0$.*

*Proof of Proposition* 2. (i) For each $d \in \mathbb{N}$, let $U_d$ be the collection of $\xi \in \mathbb{C}$ such that $\omega_d^*(\xi) = \infty$. Then $U_d$ is $\mathbf{\Pi}_2^0$, since

$$\begin{aligned}
\xi \in U_d &\Leftrightarrow \omega_d^*(\xi) = \infty \\
&\Leftrightarrow \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ (\omega_d^*(\xi, b+c) > a) \\
&\Leftrightarrow \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ \exists \alpha \in A_{d, b+c} \left(0 < |\xi - \alpha| < \frac{1}{(b+c)^{ad+1}}\right) \\
&\Leftrightarrow \xi \in \bigcap_{a\in\mathbb{N}} \bigcap_{b\in\mathbb{N}} \bigcup_{c\in\mathbb{N}} \bigcup_{\alpha\in A_{d, b+c}} V(a, b, c, \alpha),
\end{aligned}$$

where $V(a, b, c, \alpha)$ is the collection of $\xi \in \mathbb{C}$ such that $0 < |\xi - \alpha| < 1/(b+c)^{ad+1}$, which is open. Since it is easy to see that for each $d$, $\omega_d^*(\xi) = \infty$ implies $\omega_{d+1}^*(\xi) = \infty$, we have $U = \bigcup_{d=1}^{\infty} U_d$ and $U$ is $\mathbf{\Sigma}_3^0$. It is well known that if $D$ is a countable dense set in a perfect Polish space, then $D$ is $\mathbf{\Sigma}_2^0$-complete. Thus, by Lemma 1, $A$ is $\mathbf{\Sigma}_2^0$-complete.

(ii) By definition, $T$ is the collection of $\xi \in \mathbb{C}$ such that $\omega^*(\xi) = \infty$ and $\forall a \in \mathbb{N}\ \ (\omega_a^*(\xi) < \infty)$. Thus, $T = M \cap N$, where $M = \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty\}$

and $N = \{\xi \in \mathbb{C} : \forall \alpha \in \mathbb{N} \ \ (\omega_a^*(\xi) < \infty)\}$. Now $M$ is $\mathbf{\Pi}_4^0$, since

$$\begin{aligned}\xi \in M &\Leftrightarrow \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ (\omega_{b+c}^*(\xi) > a)\\ &\Leftrightarrow \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ \exists d \in \mathbb{N}\ \forall e \in \mathbb{N}\ \exists f \in \mathbb{N}\\ &\qquad \left(\omega_{b+c}^*(\xi, e+f) > a + \frac{1}{d+1}\right)\\ &\Leftrightarrow \xi \in \bigcap_{a\in\mathbb{N}} \bigcap_{b\in\mathbb{N}} \bigcup_{c\in\mathbb{N}} \bigcup_{d\in\mathbb{N}} \bigcap_{e\in\mathbb{N}} \bigcup_{f\in\mathbb{N}} W(a, b, c, d, e, f),\end{aligned}$$

where $W(a, b, c, d, e, f)$ is the collection of $\xi \in \mathbb{C}$ such that $\omega_{b+c}^*(\xi, e+f) > a + 1/(d+1)$, which is open by the argument above. So $N$ is $\mathbf{\Pi}_3^0$, since by (i) $U$ is $\mathbf{\Sigma}_3^0$ and

$$\begin{aligned}\xi \in N &\Leftrightarrow \forall a \in \mathbb{N}\ (\omega_a^*(\xi) < \infty)\\ &\Leftrightarrow \xi \in \mathbb{C} - U.\end{aligned}$$

Hence $T$ is $\mathbf{\Pi}_4^0$, being the intersection of two $\mathbf{\Pi}_4^0$ sets. Since $\xi \in S \Leftrightarrow \xi \notin T, \xi \notin U$, and $\xi \notin A$, $S$ is $\mathbf{\Sigma}_4^0$. □

In $2^{\mathbb{N}}$, $Q$ is the collection of sequences which end in zeros.

**Lemma 3.** *There exists a continuous function $\nu$ from $2^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ such that*
(i) *for each $d \in \mathbb{N}$, $\alpha \in 2^{\mathbb{N}}$, $\nu(\alpha)(d) \leq \nu(\alpha)(d+1)$;*
(ii) $\alpha \in Q \Leftrightarrow \lim_{d\to\infty} \nu(\alpha)(d) < \infty$.

*Proof of Lemma* 3. Let $\alpha \in 2^{\mathbb{N}}$. We produce $\beta = \nu(\alpha)$ recursively. First $\beta(1) = \alpha(1)$. Suppose that we have defined $\beta(i)$ for all $i \leq k$. Put $\beta(k+1) = \beta(k)$ if $\alpha(k+1) = 0$ and $\beta(k+1) = \beta(k)+1$ otherwise. It is easy to see that the function $\nu$ satisfies (i). As long as $\alpha$ ends in zeros, so does $\nu(\alpha)$ in constants. Otherwise, $\nu(\alpha)(d)$ goes to infinity as $d \to \infty$, because for infinitely many $d$'s, $\nu(\alpha)(d+1) = \nu(\alpha)(d) + 1$. So (ii) is valid. For given $d \in \mathbb{N}$, $\alpha_1, \alpha_2 \in 2^{\mathbb{N}}$, such that $\alpha_1(i) = \alpha_2(i)$ for all $i \leq d$, $\nu(\alpha_1)(i) = \nu(\alpha_2)(i)$ for all $i \leq d$. So $\nu$ is continuous. This completes Lemma 3. □

From Lemma 3, $\alpha \notin Q \Leftrightarrow \lim_{d\to\infty} \nu(\alpha)(d) = \infty$. To prove our main theorem, we need a standard example of the $\mathbf{\Pi}_3^0$-complete set.

**Lemma 4.** *The set $P_3 = \{\alpha = (\alpha_d) \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall d \in \mathbb{N}\ (\alpha_d \in Q)\}$ is $\mathbf{\Pi}_3^0$-complete.*
*Proof.* See [2].

The following theorem is the main result of the paper.

**Theorem 5.** *There is a continuous function $f$ from $(2^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{C}$ such that*

$$\alpha \in P_3 \Leftrightarrow f(\alpha) \in T \quad \textit{and} \quad \alpha \notin P_3 \Leftrightarrow f(\alpha) \in U.$$

*In particular, $T$ is $\mathbf{\Pi}_3^0$-hard and $U$ is $\mathbf{\Sigma}_3^0$-complete.*

Roughly speaking, the original statement of a theorem of Schmidt is the following: Let $\alpha_1, \alpha_2, \ldots$ be any nonzero algebraic numbers and let $\nu_1, \nu_2, \ldots$ be any real numbers exceeding 1. Then we may find $\xi \in \mathbb{C}$ such that according to $\alpha_1, \alpha_2, \ldots$ and $\nu_1, \nu_2, \ldots$, $\xi$ is a $U$ number or $T$ number.

By using $\nu$, which is constructed in Lemma 3, we shall effectively control $\nu_i$'s so that we are able to prove Theorem 5. In order to make it work, we need to state the reformulated version of a theorem of Schmidt which will play a crucial role in the proof of Theorem 5.

**Theorem S** (Schmidt). *There exists a sequence* $\langle S_n \rangle$ *such that for each* $n \in \mathbb{N}$,

(i) $S_n$ *is a function from* $\mathbb{A}^n \times \mathbb{P}^n$ *to* $\mathbb{A}^n \times (0, 1)^n$ *and* $S_{n+1}|_n = S_n$.

(ii) *Suppose that*

$$S_n((\theta_1, \dots, \theta_n), (\nu_1, \dots, \nu_n)) = ((\gamma_1, \dots, \gamma_n), (\lambda_1, \dots, \lambda_n)).$$

*Then for each* $j < n$, $\gamma_j/\theta_j$ *is rational,* $H_{j+1} > 2H_j$ *and* $\frac{1}{4}H_j^{-1} < \gamma_{j+1} - \gamma_j < \frac{1}{2}H_j^{-1}$, *where* $H_j = h_j^{\nu_j}$ *and* $h_j = \mathrm{ht}(\gamma_j)$, *and furthermore, we have* $|\gamma_j - \beta| > B^{-1}$ *for all algebraic numbers* $\beta$ *with degree* $d \le j$ *distinct from* $\gamma_1, \dots, \gamma_j$, *where* $B = \lambda_d^{-1} b^{(3d)^4}$ *and* $b$ *denotes the height of* $\beta$.

*Proof.* See [1, pp. 85–94].

Using Theorem S we define the function $S^*$ from $\mathbb{A}^{\mathbb{N}} \times \mathbb{P}^{\mathbb{N}}$ to $\mathbb{A}^{\mathbb{N}} \times (0, 1)^{\mathbb{N}}$ as follows: $S^*((\theta_1, \theta_2, \dots), (\nu_1, \nu_2, \dots)) = ((\gamma_1, \gamma_2, \dots)(\lambda_1, \lambda_2, \dots))$, where for each $n$, $S_n((\theta_1, \dots, \theta_n), (\nu_1, \dots, \nu_n)) = ((\gamma_1, \dots, \gamma_n), (\lambda_1, \dots, \lambda_n))$. $S^*$ is well defined by Theorem S(i).

*Proof of Theorem* 5. Let $\alpha \in (2^{\mathbb{N}})^{\mathbb{N}}$. Fix a bijection $\langle\ ,\ \rangle$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For each $d, k \in \mathbb{N}$, define

$$\nu_{\langle d, k\rangle} = (\nu(\alpha_d)(k) + 1)(3d)^5 \quad \text{and} \quad \theta_{\langle d, k\rangle} = \theta_{d,k},$$

where the function $\nu$ is constructed in Lemma 3. Put $\mathbb{A} = \{\theta_{d,k}\}$ and $\deg(\theta_{d,k}) = d$. Say

$$S^*((\theta_1, \theta_2, \dots), (\nu_1, \nu_2, \dots)) = ((\gamma_1, \gamma_2, \dots), (\lambda_1, \lambda_2, \dots)).$$

Then by Theorem S(ii), $\gamma_1, \gamma_2, \dots$ tends to a limit $\xi$ which is a real number and satisfies

(1) $\quad |\xi - \beta| \ge B^{-1}$ for all algebraic numbers $\beta$ distinct from $\gamma_1, \gamma_2, \dots$,

and also

$$\tfrac{1}{4}H_j^{-1} \le \xi - \gamma_j \le H_j^{-1} \quad \text{for all } j. \tag{2}$$

Define

$$f(\alpha) = \lim_{j\to\infty} \gamma_j = \xi.$$

*Claim.* $f$ is continuous from $(2^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{C}$.

*Proof of the claim.* Suppose $(\alpha_d^{(m)}) \to (\alpha_d)$ as $m \to \infty$, where for each $m$, $(\alpha_d^{(m)}) \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $(\alpha_d) \in (2^{\mathbb{N}})^{\mathbb{N}}$. Say for each $m$,

$$f((\alpha_d^{(m)})) = \xi_m = \lim_{k\to\infty} \gamma_k^{(m)} \quad \text{and} \quad f((\alpha_d)) = \xi = \lim_{k\to\infty} \gamma_k,$$

where for each $k \in \mathbb{N}$, $\gamma_k^{(m)}$ and $\gamma_k$ are defined by $S^*$, according to $(\alpha_d^{(m)})$ and $(\alpha_d)$. Let $\varepsilon > 0$. Choose $a_0$ such that $1/2^{a_0-2} < \varepsilon$. Since $(\alpha_d^{(m)})$ goes to $(\alpha_d)$ as $m \to \infty$, by the definition of $\gamma_k^{(m)}$ and $\gamma_k$ we may find $N_0 \in \mathbb{N}$ such that $|\gamma_{a_0}^{(m)} - \gamma_{a_0}| = 0$ for all $m \ge N_0$. Then for all $m \ge N_0$, we have the following inequality:

$$|\xi_m - \xi| \le |\xi_m - \gamma_{a_0}^{(m)}| + |\gamma_{a_0}^{(m)} - \gamma_{a_0}| + |\gamma_{a_0} - \xi| < \frac{1}{2^{a_0-2}} < \varepsilon,$$

since from (2) and Theorem S(ii),

$$|\xi_m - \gamma_a^{(m)}| \leq (H_a^{(m)})^{-1} < \frac{1}{2^{a-1}}(H_1^{(m)})^{-1} \leq \frac{1}{2^{a-1}}$$

and

$$|\xi - \gamma_a| \leq H_a^{-1} < \frac{1}{2^{a-1}} H_1^{-1} \leq \frac{1}{2^{a-1}}$$

for all $a \geq 1$. So $f$ is a continuous function. □

Now we show the main part of the theorem. Depending on the properties of $\nu$, Theorem S guarantees that we produce a $T$ number or $U$ number. So we divide the following two cases so that one can have more intuitive ideas.

*Case* 1. $\alpha = (\alpha_d) \notin P_3$, i.e., $\exists d \in \mathbb{N}\ (\alpha_d \notin Q)$.

Fix such $d$, i.e., $\alpha_d \notin \mathbb{Q}$. Then by Lemma 3, we have

$$\lim_{k\to\infty} (\nu(\alpha_d)(k) + 1) = \infty .$$

It is clear that for all $k$, $h = h_{\langle d,k\rangle}$,

$h^{-d\omega_d^*(\xi,h)-1} \leq |\xi - \gamma_{\langle d,k\rangle}| \leq h^{-\nu\langle d,k\rangle}$ from (2) and the definition of $\omega_d^*(\xi, h)$,

where $f(\alpha) = \xi$. So $d\omega_d^*(\xi, h_{\langle d,h\rangle}) \geq \nu_{\langle d,k\rangle} - 1$, i.e.,

$$(3) \qquad \omega_d^*(\xi, h_{\langle d,k\rangle}) \geq \frac{\nu_{\langle d,k\rangle} - 1}{d} \geq (\nu(\alpha_d)(k) + 1)3^5 d^4 - \frac{1}{d} \quad \text{for all } k .$$

It is easy to see that $\limsup_{k\to\infty} h_{\langle d,k\rangle} = \infty$, since the right side of (3) goes to infinity as $k \to \infty$. This shows that we may choose $\{k_m\}$ such that $k_m \to \infty$ and $h_{\langle d,k_m\rangle} \to \infty$ as $m \to \infty$. From (3) we get the following inequality:

$$\begin{aligned}\omega_d^*(\xi) &= \limsup_{h\to\infty} \omega_d^*(\xi, h) \geq \limsup_{m\to\infty} \omega_d^*(\xi, h_{\langle d,k_m\rangle}) \\ &\geq \lim_{m\to\infty} (\nu(\alpha_d)(k_m) + 1)3^5 d^4 - \frac{1}{d} = \infty .\end{aligned}$$

Therefore, $\omega_d^*(\xi) = \infty$ and $f(\alpha) = \xi \in U$. So we derive $\alpha \notin P_3 \Rightarrow f(\alpha) = \xi \in U$.

*Case* 2. $\alpha = (\alpha_d) \in P_3$, i.e., $\forall d \in \mathbb{N}\ (\alpha_d \in Q)$.

Fix $d \in \mathbb{N}$. Then for all $h, k, m$, we have

$$(4) \qquad \begin{aligned} \xi - \gamma_{\langle m,k\rangle} &\geq \frac{1}{4} h_{\langle m,k\rangle}^{-(\nu(\alpha_m)(k)+1)(3m)^5}, \\ |\xi - \beta| &\geq \lambda_{\deg(\beta)}(\mathrm{ht}(\beta))^{-(3\deg(\beta))^4} \end{aligned}$$

for all algebraic numbers $\beta$ distinct from $\gamma_1, \gamma_2, \ldots$ from (1) and (2), where $f(\alpha) = \xi$. In fact, all nonzero algebraic numbers appear in these two inequalities. Let $h$ be a given natural number. Then from (4) and the definition of $\omega_d^*(\xi, h)$, we have the following inequality:

$$(5) \qquad h^{-d\omega_d^*(\xi,h)} \geq \min\{\tfrac{1}{4} h^{-M_0(3d)^5}, \lambda(d) h^{-(3d)^4}\},$$

where $M_0 = \sup\{\nu(\alpha_s)(k)+1 : s \leq d \text{ and } k < \infty\}$ and $\lambda(d) = \min\{\lambda_s : s \leq d\}$. Even if for $s \leq d$, there is no $k$ such that $h_{\langle s,k\rangle} = h$, this inequality can be applied. The value $\lambda(d)$ is positive and $1 \leq M_0 < \infty$, since $\{\lambda_s : s \leq d\}$

is the finite set of positive values and by assumption and Lemma 3, $\forall d \in \mathbb{N}$ $(\lim_{k\to\infty} \nu(\alpha_d)(k) < \infty)$. So from (5) we get

$$\omega_d^*(\xi, h) \leq \max\left\{\frac{\log 4}{\log h} + 3^5 M_0 d^4, \frac{\log 1/\lambda(d)}{d \log h} + 3^5 d^4\right\} < \infty$$

and

$$\omega_d^*(\xi) = \limsup_{h\to\infty} \omega_d^*(\xi, h) \leq \max\{3^5 M_0 d^4, 3^5 d^4\} = 3^5 M_0 d^4 < \infty.$$

Hence we can see that the inequality

$$\omega_d^*(\xi) = \limsup_{h\to\infty} \omega_d^*(\xi, h) < \infty \tag{6}$$

holds for all $d$. But for all $d$, $k$, we obtain

$$\omega_d^*(\xi, h_{\langle d,k\rangle}) \geq \frac{\nu_{\langle d,k\rangle} - 1}{d} \geq (\nu(\alpha_d)(k) + 1)3^5 d^4 - \frac{1}{d}.$$

As in Case 1, $\omega_d^*(\xi) \geq 3^5 d^4 M_1 - \frac{1}{d}$, where $M_1 = \lim_{k\to\infty} \nu(\alpha_s)(k) + 1 \geq 1$. Therefore,

$$\omega_d^*(\xi) \geq (3d)^4 \quad \text{and} \quad \omega^*(\xi) = \limsup_{d\to\infty} \omega_d^*(\xi) = \infty. \tag{7}$$

From (6) and (7), for all $d \in \mathbb{N}$, $\omega_d^*(\xi) < \infty$ and $\omega^*(\xi) = \infty$, i.e., $f(\alpha) = \xi \in T$. So we derive $\alpha \in P_3 \Rightarrow f(\alpha) = \xi \in T$.

By Case 1 and Case 2, we obtain $\alpha \in P_3 \Rightarrow f(\alpha) \in T$ and $\alpha \notin P_3 \Rightarrow f(\alpha) \in U$. By definition of $T$, $U$, it is easy to see that they are disjoint. So the continuous function $f$ satisfies $P_3 = f^{-1}(T)$ and $\mathbb{C} - P_3 = f^{-1}(U)$. This fact implies that $T$, $U$ are $\mathbf{\Pi}_3^0$-hard, $\mathbf{\Sigma}_3^0$-complete respectively, since by Lemma 4, $P_3$ is $\mathbf{\Pi}_3^0$-complete. We complete the proof of Theorem 5. □

*Remark.* We conjecture that $S$, $T$ are $\mathbf{\Sigma}_4^0$-complete, $\mathbf{\Pi}_4^0$-complete, respectively.

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## References

1. A. Baker, *Transcendental number theory*, Cambridge Univ. Press, London and New York, 1975.
2. Alexander S. Kechris, *Classical descriptive set theory*, Springer-Verlag, Berlin and New York, 1995.
3. F. Kasch and B. Vollkmann, *Zur Mahlerschen Vermutung uber S-Zahlen*, Math. Ann. **136** (1958), 442–453.
4. Haseo Ki and Tom Linton, *Normal numbers and subsets of* $\mathbb{N}$ *with given densities*, Fund. Math. (2) **144** (1994), 163–179.
5. Tom Linton, *The H sets in the unit circle are properly* $G_{\delta\sigma}$, Real Analysis Exchange **19** (1993–94), 203–211.

6. K. Mahler, *Zur approximation der exponential function un des logarithmus* I, J. Reine Angew. Math. **166** (1932), 118–136.

7. W. Morgan, C. E. M. Pearce, and A. D. Pollington, *T-numbers from an* $M_0$ *set*, Mathematika **39** (1992), 18–24.

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