PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 10, October 1995

A REMARK ON THE HAUSDORFF-YOUNG INEQUALITY

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(Communicated by J. Marshall Ash)

ABSTRACT. We shall prove a sharp Hausdorff-Young inequality of Beckner type for functions on $\mathbb{T}$ with small support.

Let $L^p(\mathbb{R})$ denote the space of complex-valued L^p functions on $\mathbb{R}$. For $1 \le p < \infty$ and $f \in L^p(\mathbb{R})$ we set

$$\|f\|_p = \left(\int_{\mathbf{R}} |f(x)|^p \, dx\right)^{1/p}.$$

For $f \in L^1(\mathbb{R})$ we define the Fourier transform by setting

$$\hat{f}(x) = \int_{\mathbf{R}} e^{-ixt} f(t) \, dt, \qquad x \in \mathbb{R}.$$

The sharp Hausdorff-Young inequality of Babenko [2] and Beckner [3] states that

$$\|\hat{f}\|_{p'} \le (2\pi)^{1/p'} B_p \|f\|_p, \qquad 1 \le p \le 2, \tag{1}$$

where $1/p + 1/p' = 1$ and $B_p = (p^{1/p}/p'^{1/p'})^{1/2}$.

We shall also consider the corresponding inequality on $\mathbb{T}$. For $g \in L^p(\mathbb{T}) = L^p(-\pi, \pi)$ we set

$$\|g\|_{L^p(\mathbf{T})} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^p \, dx\right)^{1/p}, \qquad 1 \le p < \infty,$$

and define Fourier coefficients by setting

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} g(x) \, dx, \qquad n \in \mathbb{Z}.$$

Also let $c = (c_n)_{-\infty}^{\infty}$ and define norms

$$\|c\|_q = \left(\sum_n |c_n|^q\right)^{1/q}, \qquad 1 \le q < \infty,$$

and

$$\|c\|_\infty = \sup_n |c_n|.$$

Received by the editors March 21, 1994.
1991 *Mathematics Subject Classification.* Primary 42A16.

We then have the Hausdorff-Young inequality

$$\|c\|_{p'} \leq \|g\|_{L^p(\mathbb{T})}, \qquad 1 \leq p \leq 2,$$

and this inequality is sharp. The purpose of this note is to prove a Beckner type Hausdorff-Young inequality for $\mathbb{T}$. We shall do this by considering functions on $\mathbb{T}$ with small support. We therefore set

$$H_p = \lim_{a \to 0} \sup \left\{ \frac{\|c\|_{p'}}{\|g\|_{L^p(\mathbb{T})}} ; \, g \in L^p(\mathbb{T}), \, \operatorname{supp} g \subset [-a, a], \, \|g\|_{L^p(\mathbb{T})} \neq 0 \right\},$$

where $1 \leq p \leq 2$. M. E. Andersson [1] has proved that $H_p \geq B_p$, $1 \leq p \leq 2$, and that $H_p = B_p$ if p' is an even integer. We shall here prove the following result.

Theorem. $H_p = B_p$ *for* $1 \leq p \leq 2$.

We have to prove the inequality $H_p \leq B_p$ and to do this we shall first introduce an auxiliary function φ. We shall use the Féjer kernel

$$K(x) = \frac{1}{2\pi} \left(\frac{\sin x/2}{x/2} \right)^2, \qquad x \in \mathbb{R},$$

which has Fourier transform

$$\widehat{K}(\xi) = (1 - |\xi|)\chi(\xi),$$

where χ denotes the characteristic function for the interval $[-1, 1]$. We also set $K_\delta(x) = \delta K(\delta x)$, $\delta > 0$. Then

$$\widehat{K}_\delta(\xi) = \left(1 - \frac{|\xi|}{\delta}\right) \chi_\delta(\xi),$$

where χ_δ is the characteristic function for the interval $[-\delta, \delta]$. For $0 < \delta < 1$ it then follows that

$$\widehat{K}(\xi) - \delta \widehat{K}_\delta(\xi) = 1 - \delta, \qquad |\xi| \leq \delta,$$

and that the continuous function $\widehat{K} - \delta \widehat{K}_\delta$ vanishes for $|\xi| > 1$ and is linear in the intervals $[\delta, 1]$ and $[-1, -\delta]$. We then set

$$\psi(\xi) = \frac{1}{1 - \delta}(\widehat{K}(\xi) - \delta \widehat{K}_\delta(\xi))$$

so that $\psi(\xi) = 1$ for $|\xi| \leq \delta$, $\psi(\xi)$ vanishes for $|\xi| > 1$ and is linear in the above two intervals. We also have

$$\hat{\psi}(x) = \frac{1}{1 - \delta}(\hat{\hat{K}}(x) - \delta \hat{\hat{K}}_\delta(x)) = \frac{2\pi}{1 - \delta}(K(x) - \delta K_\delta(x))$$

and

$$\int |\hat{\psi}| \, dx \leq \frac{2\pi}{1 - \delta} \left(\int |K| \, dx + \delta \int |K_\delta| \, dx \right) = 2\pi \frac{1 + \delta}{1 - \delta}.$$

We then define the function φ by setting $\varphi(x) = \psi(\delta x)$ so that $\varphi(x) = 1$ for $|x| \leq 1$. Then

$$\hat{\varphi}(x) = \frac{1}{\delta} \hat{\psi} \left(\frac{x}{\delta} \right)$$

and

$$\int |\hat{\varphi}|\,dx = \int |\hat{\psi}|\,dx \le 2\pi \frac{1+\delta}{1-\delta} = 2\pi c_\delta\,,$$

where $c_\delta = (1+\delta)/(1-\delta)$.

We then fix δ.

We also set $\varphi_a(x) = \varphi(x/a)$, $0 < a < 1$, so that $\varphi_a(x) = 1$ for $|x| \le a$. Then $\hat{\varphi}_a(x) = a\hat{\varphi}(ax)$ and

$$\int |\hat{\varphi}_a|\,dx \le 2\pi c_\delta\,. \tag{2}$$

Now, assume that $f \in L^p(\mathbb{R})$, $1 < p \le 2$, and that $f(x) = 0$ for $|x| > a$. Then $f = \varphi_a f$ and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx = \frac{1}{2\pi}\hat{f}(n)\,, \qquad n \in \mathbb{Z}.$$

It follows that

$$c_n = (2\pi)^{-2}\hat{\varphi}_a * \hat{f}(n) = (2\pi)^{-2}\int \hat{f}(n-t)\hat{\varphi}_a(t)\,dt\,,$$

and invoking Hölder's inequality one obtains

$$\begin{aligned}
|c_n|^{p'} &\le (2\pi)^{-2p'}\left(\int |\hat{f}(n-t)|\,|\hat{\varphi}_a(t)|^{1/p'}|\hat{\varphi}_a(t)|^{1/p}\,dt\right)^{p'} \\
&\le (2\pi)^{-2p'}\left(\int |\hat{f}(n-t)|^{p'}|\hat{\varphi}_a(t)|\,dt\right)\left(\int |\hat{\varphi}_a(t)|\,dt\right)^{p'/p} \\
&= (2\pi)^{-2p'}\left(\int |\hat{f}(t)|^{p'}|\hat{\varphi}_a(n-t)|\,dt\right)\left(\int |\hat{\varphi}|\,dt\right)^{p'/p}.
\end{aligned}$$

Now choose $\varepsilon > 0$. If a is small, we then get

$$\begin{aligned}
\sum_n |c_n|^{p'} &\le (2\pi)^{-2p'}\left(\int |\hat{f}(t)|^{p'}\left(\sum_n a|\hat{\varphi}(an-at)|\right)dt\right)\left(\int |\hat{\varphi}|\,dt\right)^{p'/p} \\
&\le (2\pi)^{-2p'}\left(\int |\hat{\varphi}|\,dt + \varepsilon\right)^{1+p'/p}\left(\int |\hat{f}(t)|^{p'}\,dt\right).
\end{aligned}$$

Invoking (1) and (2), we then obtain

$$\begin{aligned}
\|c\|_{p'} &\le (2\pi)^{-2}\left(\int |\hat{\varphi}|\,dt + \varepsilon\right)\|\hat{f}\|_{p'} \\
&\le (2\pi)^{-2}2\pi(c_\delta + \varepsilon)(2\pi)^{1/p'}B_p\|f\|_p \\
&= (2\pi)^{-1+1/p'}(c_\delta+\varepsilon)B_p\left(\int |f|^p\,dx\right)^{1/p} \\
&= (c_\delta+\varepsilon)B_p\left(\frac{1}{2\pi}\int |f|^p\,dx\right)^{1/p}.
\end{aligned}$$

Choosing δ small we can get the value of c_δ close to 1 and therefore we have

$$\|c\|_{p'} \le (1+2\varepsilon)B_p\|f\|_{L^p(\mathbf{T})}$$

if a is small. It follows that $H_p \le B_p$, and the proof of the theorem is complete.

References

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