BANACH SPACES OF POLYNOMIALS
WITHOUT COPIES OF $l^1$

MANUEL VALDIVIA

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Abstract. Let $X$ be a Banach space. For a positive integer $m$, let $\mathcal{P}_{w^*}(mX^*)$ denote the Banach space formed by all $m$-homogeneous polynomials defined on $X^*$ whose restrictions to the closed unit ball $B(X^*)$ of $X^*$ are continuous for the weak-star topology. For each one of such polynomials, its norm will be the supremum of the absolute value in $B(X^*)$. In this paper the bidual of $\mathcal{P}_{w^*}(mX^*)$ is constructed when this space does not contain a copy of $l^1$. It is also shown that, whenever $X$ is an Asplund space, $\mathcal{P}_{w^*}(mX^*)$ is also Asplund.

Unless stated, all linear spaces used here throughout are assumed to be non-trivial and defined over the field $\mathbb{C}$ of complex numbers. Our topological spaces will all be Hausdorff.

If $X$ is a Banach space, $X^*$ and $X^{**}$ will be its conjugate and second conjugate, respectively. We identify $X$ in the usual manner with a subspace of $X^{**}$. $B(X)$ is the closed unit ball of $X$. The duality between $X$ and $X^*$ is denoted by $\langle \cdot , \cdot \rangle$, i.e., for $x$ in $X$ and $u$ in $X^*$, $\langle x, u \rangle = u(x)$. The norm of any Banach space will be represented by $\| \cdot \|$. In the product $X_1 \times X_2 \times \cdots \times X_m$ of the Banach spaces $X_1, X_2, \ldots, X_m$ we consider the norm given by the Minkowski functional of $B(X_1) \times B(X_2) \times \cdots \times B(X_m)$. By $\mathcal{M}(X_1, X_2, \ldots, X_m)$ we denote the linear space over $\mathbb{C}$ of the continuous $m$-linear forms defined on $X_1 \times X_2 \times \cdots \times X_m$. We assume $\mathcal{M}(X_1, X_2, \ldots, X_m)$ provided with the usual norm, that is, for any such $m$-linear form $f$,

$$\| f \| := \sup \{|f(x_1, x_2, \ldots, x_m)|: (x_j) \in B(X_j), j = 1, 2, \ldots, m\}.$$ 

$\mathcal{M}_{w^*}(X_1^*, X_2^*, \ldots, X_m^*)$ is the subspace of $\mathcal{M}(X_1^*, X_2^*, \ldots, X_m^*)$ formed by those elements whose restrictions to $B(X_1^*) \times B(X_2^*) \times \cdots \times B(X_m^*)$ are continuous with respect to the topology induced by the weak-star topology of $X_1^* \times X_2^* \times \cdots \times X_m^*$.

For a Banach space $X$ and a positive integer $m$, $\mathcal{P}(mX)$ is the linear space of the continuous $m$-homogeneous polynomials defined on $X$. We consider $\mathcal{P}(mX)$ endowed with the usual norm, i.e., for any such $f$,

$$\|f\| := \sup \{|f(x)|: x \in B(X)\}.$$
\( \mathcal{P}_w^*(mX^*) \) represents the subspace of \( \mathcal{P}(mX^*) \) whose elements are those polynomials that are weak-star continuous in \( B(X^*) \). \( \mathcal{P}_w^*(mX^*) \) is the Banach subspace of \( \mathcal{P}(mX^*) \) algebraically defined as the closure of \( \mathcal{P}_w^*(mX^*) \) in \( \mathcal{P}(mX^*) \) when this space is endowed with the compact open topology, i.e., the topology of uniform convergence on compact subsets of \( X^* \).

A Banach space \( X \) is said to be Asplund if every separable subspace \( Y \) of \( X \) has separable dual \( Y^* \) or, equivalently, \( X^* \) has the Radon-Nikodym property.

For a subset \( \{x_j: j \in J\} \) of a Banach space \( X \), \( \text{lin}\{x_j: j \in J\} \) denotes its linear span while \( [x_j: j \in J] \) is its closed linear span.

In a Banach space \( X \), a biorthogonal system
\[
(x_j, u_j)_{j \in J}, \quad x_j \in X, \ u_j \in X^*, \quad \langle x_j, u_j \rangle = 1, \quad \langle x_j, u_h \rangle = 0, \quad j \neq h, \quad j, \ h \in J,
\]
is a Markushevich basis if \( \{x_j: j \in J\} \) coincides with \( X \) and \( \text{lin}\{u_j: j \in J\} \) is weak-star dense in \( X^* \).

If \( S \) is a compact topological space, \( C(S) \) is the real vector space of the continuous real-valued functions defined on \( S \) with the usual norm. \( S \) is said to be Corson if it is homeomorphic to a subspace \( T \) of the product \( \mathbb{R}^J \), for some \( J \) depending on \( S \), where \( \mathbb{R} \) is the set of reals equipped with the usual topology, such that if the point \((a_j: j \in J)\) is in \( T \), then the set \( \{j \in J: a_j \neq 0\} \) is countable. \( S \) is an Eberlein compact if it is homeomorphic to a weakly compact subset of a Banach space. Using a result of Amir and Lindenstrauss [1], if \( S \) is Eberlein, then it is homeomorphic to a weakly compact subset \( A \) of the real space \( c_0(J) \), for some index set \( J \) depending on \( S \); clearly, the mapping \( \varphi \) from \( A \) to \( \mathbb{R}^J \) which assigns to each element \((a_j: j \in J)\) in \( A \) the element \((a_j: j \in J)\) of \( \mathbb{R}^J \) is a homeomorphism from \( A \) onto \( \varphi(A) \) with \( \{j \in J: a_j \neq 0\} \) countable, hence \( S \) is a Corson compact.

We have shown two results, in [4] and [5], respectively, that are more general than the following: (a) If \( X \) is an Asplund space admitting a Markushevich basis \((x_j, u_j)_{j \in J}\) such that, for each \( u \) in \( X^* \), the set \( \{j \in J: \langle x_j, u \rangle \neq 0\} \) is countable, then \( X \) is weakly compactly generated. (b) If \( S \) is a Corson compact and \( E \) is a subspace of \( C(S) \) that is closed for the topology of pointwise convergence, there is a Markushevich basis \((f_j, u_j)_{j \in J}\) for \( E \) such that, for each \( s \) in \( S \), the set \( \{j \in J: f_j(s) \neq 0\} \) is countable.

Let \( S \) and \( T \) be two topological spaces. Let \( \varphi \) be a set-valued map from \( S \) to \( T \). A mapping \( \psi \) from \( S \) to \( T \) is said to be a selector of \( \varphi \) whenever \( \psi(s) \) belongs to \( \varphi(s) \) for each \( s \) in \( S \). The mapping \( \varphi \) is upper semicontinuous provided there is, for each \( s \) in \( S \) and each open neighborhood \( V \) of \( \varphi(s) \), a neighborhood \( U \) of \( s \) such that \( \varphi(x) \) is contained in \( V \) for every \( x \) in \( U \).

We shall need the following result [3]: (c) Let \( S \) be a metric space and let \( Y \) be a Banach space such that \( Y^* \) has the Radon-Nikodym property. Let \( \varphi \) be a set-valued map from \( S \) to \( Y^* \). If \( \varphi \) is upper semicontinuous and \( \varphi(x) \) is a non-empty compact set for each \( x \) in \( X \) with respect to the weak-star topology of \( Y^* \), then there is a selector \( \psi \) of \( \varphi \) of the first Baire class between the metric spaces \( S \) and \( Y^* \).

We say that a mapping \( f \) between the topological spaces \( S \) and \( T \) is quasi-Baire if there is a countable set \( L \) of continuous mappings from \( S \) to \( T \) such that \( f \) belongs to the closure of \( L \) in the topological space \( T^S \).

The following result that we have shown in [6] will also be needed: (d) Let \( X \) be a real Banach space. If there is a quasi-Baire mapping \( \Phi \) from \( X \) to \( X^* \)
such that \( \|\Phi(x)\| = 1, \langle x, \Phi(x) \rangle = \|x\|, \ x \in X, \ x \neq 0 \), then \( X \) is an Asplund space.

**Theorem 1.** If the Banach spaces \( X_1, X_2, \ldots, X_m \) are all Asplund, then \( \mathcal{M}_w(X_1^*, X_2^*, \ldots, X_m^*) \) is also Asplund.

**Proof.** In order to simplify notation, let us write

\[ Y := \mathcal{M}_w(X_1^*, X_2^*, \ldots, X_m^*) \]

and \( B \) instead of \( B(X_1^*) \times B(X_2^*) \times \cdots \times B(X_m^*) \) provided with the topology induced by the weak-star topology of \( X_1^* \times X_2^* \times \cdots \times X_m^* \).

For each \( f \) in \( Y \), we define \( \varphi(f) \) as the set of points \( x \in B \) for which \( f(x) = \|f\| \). It is plain that \( \varphi(f) \) is non-empty and compact. Also, it is not hard to see that the set-valued map \( \varphi \) from \( Y \) to \( X_1^* \times X_2^* \times \cdots \times X_m^* \) is upper semicontinuous when the latter space has the weak-star topology. Result (c) yields then a selector \( \psi \) of \( \varphi \) of the first Baire class between the metric spaces \( Y \) and \( X_1^* \times X_2^* \times \cdots \times X_m^* \). Hence, there is a sequence \( (\psi_n) \) of continuous mappings from \( Y \) to \( X_1^* \times X_2^* \times \cdots \times X_m^* \) such that, for each \( f \) in \( Y \),

\[ \lim_{n} \psi_n(f) = \psi(f). \]

For each \( f \) in \( Y \), we put \( f = f_1 + if_2 \), with \( f_1 \) and \( f_2 \) real valued,

\[ ||f'|| := \sup\{|f_1(x_1, x_2, \ldots, x_m)|: (x_1, x_2, \ldots, x_m) \in B\}. \]  

By setting \( Z := \{f_1: f \in Y\} \) we obtain a real vector space \( Z \) in which we consider the norm described in (1). If \( Y_r \) denotes the real Banach space subjacent to \( Y \) and we put \( \Gamma f = f_1, f \in Y_r \), then

\[ \Gamma: Y_r \to Z \]

is an onto linear map. Besides, if \( (x_1, x_2, \ldots, x_m) \) belongs to \( B \) and \( f \) is in \( Y \), we have

\[ f(ix_1, x_2, \ldots, x_m) = f_1(ix_1, x_2, \ldots, x_m) + if_2(ix_1, x_2, \ldots, x_m) = if_1(x_1, x_2, \ldots, x_m) - f_2(x_1, x_2, \ldots, x_m), \]

and thus

\[ ||f'|| \leq ||f|| \leq 2||f||, \]

concluding that \( \Gamma \) is a topological isomorphism from \( Y_r \) onto \( Z \). It all reduces to show that \( Z \) is an Asplund space, since then \( Y_r \), and thereby \( Y \), will also be Asplund.

Each element \( (u_1, u_2, \ldots, u_m) \) of \( X_1^* \times X_2^* \times \cdots \times X_m^* \) defines a continuous linear form \( \Lambda(u_1, u_2, \ldots, u_m) \) on \( Z \) by putting, for each \( g \) of \( Z \),

\[ (g, \Lambda(u_1, u_2, \ldots, u_m)) = g(u_1, u_2, \ldots, u_m). \]

We write

\[ \Phi := \Lambda \circ \psi \circ \Gamma^{-1}, \quad \Phi_n = \Lambda \circ \psi_n \circ \Gamma^{-1}, \quad n = 1, 2, \ldots. \]

The mappings \( \Phi_n: Z \to Z^* \) are then continuous, \( n = 1, 2, \ldots \), and

\[ \lim_n \Phi_n(g) = \Phi(g), \quad g \in Z. \]

Therefore,

\[ \Phi: Z \to Z^* \]
is a quasi-Baire map. Let us write $M := A(B)$. Clearly, $M$ is contained in $B(Z^*)$ and so, for each $f$ in $Y$, $\|\Phi(f)\| \leq 1$. Moreover, if $(u_1, u_2, \ldots, u_m)$ is the element of $B$ such that $\psi(f) = (u_1, u_2, \ldots, u_m)$, then

$$\|f_1\| \leq \|f\| = f(u_1, u_2, \ldots, u_m) = (f_1, \Phi(f_1)) \leq \|f_1\| \cdot \|\Phi(f_1)\|,$$

and we have

$$\|\Phi(f_1)\| = 1, \quad (f_1, \Phi(f_1)) = \|f_1\|, \quad f_1 \in Z, f_1 \neq 0.$$

We apply now result (d) to conclude that $Z$ is Asplund. Q.E.D.

**Corollary 1.1.** Let $m$ be a positive integer. If $X$ is an Asplund space, then $\mathcal{W} \ast (mX^*)$ is an Asplund space.

**Proof.** In the previous theorem we take $X = X_1 = X_2 = \cdots = X_m$. Then subspace $F$ of $\mathcal{W} \ast (X_1^*, X_2^*, \ldots, X_m^*)$ consisting of the symmetric $m$-linear forms is a Banach space which is Asplund. We conclude by recalling that $F$ is isomorphic to $\mathcal{W} \ast (mX^*)$. Q.E.D.

The following result can be found in [2]: (e) Let $X$ be a Banach space with no copy of $l^1$. Let $A$ be a weak-star compact subset of $X^*$. If $B$ is the closed absolutely convex hull of $A$ in $X^*$, then $B$ is weak-star compact.

**Theorem 2.** Let $X$ be a Banach space. If, for a positive integer $m$, $\mathcal{W} \ast (mX^*)$ does not contain a copy of $l^1$, then $\mathcal{W} \ast (mX^*)$ identifies with $\mathcal{W} \ast (mX^*)$.

**Proof.** Let $\lambda$ be the map from $X^*$ to $\mathcal{W} \ast (mX^*)$ such that, for each $u$ in $X^*$ and each $g$ in $\mathcal{W} \ast (mX^*)$, $(\lambda(u), g) = g(u)$. We set $A := \lambda(B(X^*))$. It can be simply checked that $A$ is a weak-star compact subset of $\mathcal{W} \ast (mX^*)$. Now, let $B$ stand for the closed absolutely convex hull of $A$ in the Banach space $\mathcal{W} \ast (mX^*)$. In light of the previously mentioned result (e), we have that $B$ is also weak-star compact. We deduce from this that, if $v$ is a non-zero weak-star continuous linear form defined on $\mathcal{W} \ast (mX^*)$, then $v \circ \lambda$ is a non-zero element of $\mathcal{W} \ast (mX^*)$. Consequently, $B$ is the closed unit ball of $\mathcal{W} \ast (mX^*)$.

Let $\Lambda$ be a map from $X^*$ to $\mathcal{W} \ast (mX^*)$ such that for each $u$ in $X^*$ and each $g$ in $\mathcal{W} \ast (mX^*)$,

$$(g, \Lambda u) = g(u).$$

In $\mathcal{W} \ast (mX^*)$, let $H$ be the linear hull of $\Lambda(X^*)$ and $E$ the closure of $H$. $E$ is a Banach space. By $\Gamma$ we denote the mapping from $E$ to $\mathcal{W} \ast (mX^*)$ which assigns to each $u$ in $E$ its restriction to $\mathcal{W} \ast (mX^*)$. We next show that $\Gamma$ is an isometry.

Obviously, $\lambda = \Gamma \circ \Lambda$. If $g$ belongs to $\mathcal{W} (mX^*)$ and $\alpha \in \mathbb{C}$, then $g(\alpha x) = \alpha^m g(x)$, and thus, if, for a positive integer $p$, $B_p(X^*)$ represents the closed unit ball in $X^*$ of radius $2^{-p}$, the closed absolutely convex hull of $\Lambda(B_p(X^*))$ in $\mathcal{W} \ast (mX^*)$ coincides with the closed ball in $E$ of radius $2^{-pm}$. Similarly, the closed absolutely convex hull of $\lambda(B_p(X^*))$ in $\mathcal{W} \ast (mX^*)$ is the closed ball in $\mathcal{W} \ast (mX^*)$ of radius $2^{-pm}$.

Now, let $P$ and $Q$ be the absolutely convex hulls of $\Lambda(B(X^*))$ and $\lambda(B(X^*))$ in $E$ and $\mathcal{W} \ast (mX^*)$, respectively. Take a non-zero element $w$ of $E$. We find $0 < \beta < 1$ such that $\|\beta w\| < 2^{-2m}$. We choose $w_1$ in $P$ so that

$$\|\beta w - 2^{-2m} w_1\| < 2^{-4m}.$$
Proceeding by induction, let us assume that, for a positive integer \( p \), we have found \( w_1, w_2, \ldots, w_p \) in \( P \) such that
\[ \| \beta w - 2^{-2m}w_1 - 2^{-4m}w_2 - \cdots - 2^{-2pm}w_p \| < 2^{-2(p+1)m}. \]

We then determine \( w_{p+1} \) in \( P \) for which
\[ \| 2^{m(p+1)}(\beta w - 2^{-2m}w_1 - 2^{-4m}w_2 - \cdots - 2^{-2pm}w_p) - w_{p+1} \| < 2^{-2m} \]
and
\[ \| \beta w - 2^{-2m}w_1 - 2^{-4m}w_2 - \cdots - 2^{-2pm}w_p - 2^{m(p+1)}w_{p+1} \| < 2^{-2(p+2)m}. \]

Hence, in \( E \), we have
\[ \beta w = \sum_{p=1}^{\infty} 2^{-2pm}w_p. \]

We take now a finite subset \( A_p \) of \( B_p(X^*) \) such that \( 2^{-pm}w_p \) is in the absolutely convex hull in \( E \) of \( \Lambda(A_p) \). If \( K \) is the closed absolutely convex hull in \( E \) of \( \bigcup_{p=1}^{\infty} \Lambda(A_p) \) we have that \( K \) is compact and \( \beta w \) is clearly in \( K \). Fix an element \( g \) of \( \mathcal{P}(mX^*) \) such that \( \langle g, w \rangle \neq 0 \). We select an element \( f \) of \( \mathcal{P}(mX^*) \) satisfying
\[ |f(x) - g(x)| < \frac{1}{2} \beta |\langle g, w \rangle|, \quad x \in \bigcup_{p=1}^{\infty} A_p. \]

Then
\[ |\langle f - g, \Lambda(x) \rangle| = |f(x) - g(x)| < \frac{1}{2} \beta |\langle g, w \rangle|, \quad x \in \bigcup_{p=1}^{\infty} A_p, \]

hence
\[ |\langle f - g, \beta w \rangle| \leq \frac{1}{2} \beta |\langle g, w \rangle|, \]

and therefore
\[ |\langle f, w \rangle| \geq |\langle f, \beta w \rangle| \geq |\langle g, \beta w \rangle| - |\langle f - g, \beta w \rangle| \geq \beta |\langle g, w \rangle| - \frac{1}{2} \beta |\langle g, w \rangle| = \frac{1}{2} \beta |\langle g, w \rangle| \neq 0, \]
thus proving \( \Gamma \) to be one-to-one.

Take now \( v \) in \( \mathcal{P}(mX^*)^* \). We proceed as before and find \( \gamma > 0 \) and \( v_p \) in \( Q, p = 1, 2, \ldots, \) such that in \( \mathcal{P}(mX^*)^* \)
\[ \gamma v = \sum_{p=1}^{\infty} 2^{-2pm}v_p. \]

It is then possible to determine a null sequence \( (x_q) \) in \( B(X^*) \) and \( \alpha_q > 0, \quad q = 1, 2, \ldots, \sum_{q=1}^{\infty} \alpha_q < 1 \), so that
\[ \gamma v = \sum_{q=1}^{\infty} \alpha_q \lambda(x_q). \]
But the series \( \sum_{q=1}^{\infty} \alpha_q \Lambda(x_q) \) converges to an element \( yz \) in \( E \). Thus
\[
\Gamma(z) = \gamma^{-1} \sum_{q=1}^{\infty} \alpha_q (\Gamma \circ \Lambda)(x_q) = \gamma^{-1} \sum_{q=1}^{\infty} \alpha_q \lambda(x_q) = v,
\]
and we have that \( \Gamma \) is onto.

Having in mind that \( \Gamma(P) = Q \), Banach’s isomorphism theorem guarantees that
\[
\Gamma: E \to \mathcal{P}_w^*(mX^*)^*
\]
is an isometry.

As a by-product of this we have that \( B(\mathcal{P}_w^*(mX^*)) \) is \( \sigma(\mathcal{P}_w^*(mX^*), E) \)-dense in \( B(\mathcal{P}_w^*(mX^*)) \), and since this set of linear forms on \( E \) is equicontinuous, it follows that in \( B(\mathcal{P}_w^*(mX^*)) \) the compact open topology of \( \mathcal{P}(mX^*) \) coincides with \( \sigma(\mathcal{P}_w^*(mX^*), E) \); thus \( B(\mathcal{P}_w^*(mX^*)) \) is compact for the topology of pointwise convergence on \( X^* \). The desired result is now immediate. Q.E.D.

**Corollary 1.2.** Let \( m \) be a positive integer. If the Banach space \( X \) is Asplund, then \( \mathcal{P}_w^*(mX^*) \) is the bidual of \( \mathcal{P}_w^*(mX^*) \).

**Proof.** It follows from Corollary 1.1 that \( \mathcal{P}_w^*(mX^*) \) is Asplund and so it does not contain copies of \( l^1 \). Our previous theorem now applies. Q.E.D.

**Theorem 3.** Let \( X \) be a Banach space such that \( X^* \) has the approximation property. If for a positive integer \( m \), \( \mathcal{P}_w^*(mX^*) \) does not contain copies of \( l^1 \), then \( \mathcal{P}_w^*(mX^*)^{**} \) coincides with \( \mathcal{P}(mX^*) \).

**Proof.** For a positive integer \( p \), we write \( \mathcal{L}_s^p(X^*) \) meaning the subspace of \( \mathcal{M}(X^*, X^*, \ldots, X^*) \) consisting of all symmetric \( p \)-linear forms. This space identifies in the usual fashion with the Banach space \( \mathcal{L}(X^*, \mathcal{L}_s^p(X^*)) \) of the bounded linear operators from \( X^* \) to \( \mathcal{L}_s^p(X^*) \), assuming \( \mathcal{L}_s^p(X^*) = C \) for \( p = 1 \).

Let \( f \) be in \( \mathcal{P}(mX^*) \). We want to show that, given \( \varepsilon > 0 \) and a compact subset \( K \) of \( X^* \), there is an element \( g \) in \( \mathcal{P}_w^*(mX^*) \) such that \( |f(x) - g(x)| < \varepsilon, \ x \in K \). To do this we follow a complete induction procedure. If \( m = 1 \), then \( f \) is an element of \( X^{**} \), so there is an element \( g \) of \( X \), hence of \( \mathcal{P}_w^*(mX^*) \), such that \( |f(x) - g(x)| < \varepsilon, \ x \in K \). We assume the property true for a positive integer \( m - 1 \) and show it still holds for \( m \). We may assume, without loss of generality, that \( K \) is contained in \( B(X^*) \). Let \( h \) be the element of \( \mathcal{L}_s^p(mX^*) \) such that \( h(x, x, \ldots, x) = f(x), \ x \in X^* \). For \( x_1 \) in \( X^* \), \( h(x_1, x, \ldots, x) = h(x_1, x_2, x_3, \ldots, x_m) = h(x_1, x_2, x_3, \ldots, x_m) \).

We get hold of an operator
\[
\zeta: X^* \to \mathcal{L}_s^p(m^{-1}X^*)
\]
of finite rank such that
\[
\|\zeta(x) - h(x, x, \ldots, x)\| < \frac{\varepsilon}{2}, \quad x \in K.
\]
We find \( h_1, h_2, \ldots, h_q \) in \( \mathcal{L}_s^p(m^{-1}X^*) \) and \( u_1, u_2, \ldots, u_q \) in \( X^{**} \) so that
\[
\zeta(x) = u_1(x)h_1 + u_2(x)h_2 + \cdots + u_q(x)h_q, \quad x \in X^*.
\]
We take now \( v_j \) in \( X, \ j = 1, 2, \ldots, q \), such that
\[
|v_j(x) - u_j(x)| \cdot \|h_j\| < \frac{\varepsilon}{4q}, \quad x \in K.
\]

By the induction hypothesis, there is a polynomial \( \kappa_j \) in \( \mathcal{P}_w (m^{-1}X^*) \) for which
\[
|h_j(x, x, \ldots, x) - \kappa_j(x)| \cdot \|v_j\| < \frac{\varepsilon}{4q}, \quad x \in K.
\]

Then
\[
g(x) := v_1(x)\kappa_1(x) + v_2(x)\kappa_2(x) + \cdots + v_q(x)\kappa_q(x), \quad x \in X^*,
\]
is an element of \( \mathcal{P}_w (mX^*) \). For each \( x \) in \( K \), we have
\[
|\zeta(x)(x, x, \ldots, x) - g(x)|
\]
\[
\leq \sum_{j=1}^q |(u_j(x) - v_j(x))h_j(x, x, \ldots, x)|
\]
\[
+ \sum_{j=1}^q |v_j(x)(h_j(x, x, \ldots, x) - \kappa_j(x))|
\]
\[
\leq \sum_{j=1}^q |u_j(x) - v_j(x)| \cdot \|h_j\| + \sum_{j=1}^q |v_j(x) - \kappa_j(x)| \cdot \|h_j(x, x, \ldots, x) - \kappa_j(x)| \leq \frac{\varepsilon}{2}.
\]

Finally, for each \( x \) in \( K \),
\[
|f(x) - g(x)| = |h(x, x, \ldots, x) - g(x)|
\]
\[
\leq |h(x, x, \ldots, x) - \zeta(x)(x, x, \ldots, x)|
\]
\[
+ |\zeta(x)(x, x, \ldots, x) - g(x)|
\]
\[
\leq \|h(x, \cdot, \ldots, \cdot) - \zeta(x)\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \text{Q.E.D.}
\]

**Corollary 1.3.** Let \( X \) be an Asplund space such that \( X^* \) has the approximation property. If \( m \) is a positive integer, \( \mathcal{P}_w (mX^*)^\circ \) coincides with \( \mathcal{P}(mX^*) \).

**Proof.** It follows immediately from Corollary 1.1 and the last theorem. \quad \text{Q.E.D.}

**Theorem 4.** Let \( X_1, X_2, \ldots, X_m \) be weakly compactly generated Asplund spaces. If \( G \) is a subspace of \( \mathcal{M}_w (X_1^*, X_2^*, \ldots, X_m^*) \), closed for the topology of pointwise convergence, then \( G \) is a weakly compactly generated Banach space.

**Proof.** Our notation will be that used in the proof of Theorem 1. For \( j = 1, 2, \ldots, m \), let \( M_j \) be a weakly compact absolutely convex subset whose linear span is dense in \( X_j \). If \( M_j^0 \) is the polar set of \( M_j \) in \( X_j^* \), then this linear space, taking \( M_j^0 \) as closed unit ball, is a normed space in which \( B(X_j^*) \) is weakly compact, hence \( B(X_j^*) \) is a Corson compact for the weak-star topology. We have then that \( B \) is a Corson compact. It can be easily seen that \( Z \) is a closed subspace of \( C(B) \) for the topology of pointwise convergence. We apply result \( \text{(b)} \) to obtain a Markushevich basis for \( \Gamma(G) \) such that, for each \( x \) in \( B \), the set \( \{ j \in J : g_j(x) \neq 0 \} \) is countable. Following the proof of Theorem 1, we have that \( Z \) is Asplund and so it contains no copy of \( l^1 \). If \( D \) is the set formed by the restrictions of the elements of \( \Lambda(B) \) to \( \Gamma(G) \), we can apply result \( \text{(e)} \) to have that the closed absolutely convex hull of \( D \) in \( \Gamma(G)^* \) is weak-star compact, and
thus it coincides with the closed unit ball of this space. From this we have that, for each \( u \) in \( \Gamma(G)^* \), \( \{ j \in J : (g_j, u) \neq 0 \} \) is countable. Result (a) assures then that \( \Gamma(G) \) is weakly compactly generated. If \( G_r \) denotes the real Banach space subjacent to \( G \), it follows that

\[ \Gamma_{|G_r} : G_r \to \Gamma(G_r) \]

is an isomorphism, and thereby \( G_r \) is weakly compactly generated. It is now immediate that \( G \) is also weakly compactly generated. Q.E.D.

**Corollary 1.4.** Let \( m \) be a positive integer. If \( X \) is a weakly compactly generated Asplund space, then \( \mathcal{R}_{w^*}(mX^*) \) is weakly compactly generated.

**Proof.** The subspace \( F \) of \( \mathcal{M}_{w^*}(X_1^*, X_2^*, \ldots, X_m^*) \) formed by the symmetric \( m \)-linear forms when \( X_1^* = X_2^* = \ldots = X_m^* = X \) is closed for the topology of pointwise convergence and so, in light of last theorem, \( F \) is weakly compactly generated. Since \( F \) is isomorphic to \( \mathcal{R}_{w^*}(mX^*) \), the conclusion follows. Q.E.D.

**References**


**Departamento de Análisis Matemático, Universidad de Valencia, Dr. Moliner, 50, 46100 Burjasot, Valencia, Spain**