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BANACH SPACES OF POLYNOMIALS WITHOUT COPIES OF l^1

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ABSTRACT. Let X be a Banach space. For a positive integer m, let $\mathscr{P}_{w^*}({}^m X^*)$ denote the Banach space formed by all m-homogeneous polynomials defined on X^* whose restrictions to the closed unit ball $B(X^*)$ of X^* are continuous for the weak-star topology. For each one of such polynomials, its norm will be the supremum of the absolute value in $B(X^*)$. In this paper the bidual of $\mathscr{P}_{w^*}({}^m X^*)$ is constructed when this space does not contain a copy of l^1. It is also shown that, whenever X is an Asplund space, $\mathscr{P}_{w^*}({}^m X^*)$ is also Asplund.

Unless stated, all linear spaces used here throughout are assumed to be nontrivial and defined over the field $\mathbb{C}$ of complex numbers. Our topological spaces will all be Hausdorff.

If X is a Banach space, X^* and X^{**} will be its conjugate and second conjugate, respectively. We identify X in the usual manner with a subspace of X^{**}. $B(X)$ is the closed unit ball of X. The duality between X and X^* is denoted by $\langle \cdot , \cdot \rangle$, i.e., for x in X and u in X^*, $\langle x , u\rangle = u(x)$. The norm of any Banach space will be represented by $\|\cdot\|$.

In the product $X_1 \times X_2 \times \cdots \times X_m$ of the Banach spaces $X_1, X_2, \dots, X_m$ we consider the norm given by the Minkowski functional of $B(X_1) \times B(X_2) \times \cdots \times B(X_m)$. By $\mathscr{M}(X_1, X_2, \dots, X_m)$ we denote the linear space over $\mathbb{C}$ of the continuous m-linear forms defined on $X_1 \times X_2 \times \cdots \times X_m$. We assume $\mathscr{M}(X_1, X_2, \dots, X_m)$ provided with the usual norm, that is, for any such m-linear form f,

$$\|f\| := \sup\{|f(x_1, x_2, \dots, x_m)| : x_j \in B(X_j),\ j = 1, 2, \dots, m\}.$$

$\mathscr{M}_{w^*}(X_1^*, X_2^*, \dots, X_m^*)$ is the subspace of $\mathscr{M}(X_1^*, X_2^*, \dots, X_m^*)$ formed by those elements whose restrictions to $B(X_1^*)\times B(X_2^*)\times\cdots\times B(X_m^*)$ are continuous with respect to the topology induced by the weak-star topology of $X_1^* \times X_2^* \times \cdots \times X_m^*$.

For a Banach space X and a positive integer m, $\mathscr{P}({}^m X)$ is the linear space of the continuous m-homogeneous polynomials defined on X. We consider $\mathscr{P}({}^m X)$ endowed with the usual norm, i.e., for any such f,

$$\|f\| := \sup\{|f(x)| : x \in B(X)\}.$$

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$\mathscr{P}_{w^*}({}^mX^*)$ represents the subspace of $\mathscr{P}({}^mX^*)$ whose elements are those polynomials that are weak-star continuous in $B(X^*)$. $\mathscr{P}_{(w^*)}({}^mX^*)$ is the Banach subspace of $\mathscr{P}({}^mX^*)$ algebraically defined as the closure of $\mathscr{P}_{w^*}({}^mX^*)$ in $\mathscr{P}({}^mX^*)$ when this space is endowed with the compact open topology, i.e., the topology of uniform convergence on compact subsets of X^*.

A Banach space X is said to be Asplund if every separable subspace Y of X has separable dual Y^* or, equivalently, X^* has the Radon-Nikodym property.

For a subset $\{x_j : j \in J\}$ of a Banach space X, $\text{lin}\{x_j : j \in J\}$ denotes its linear span while $[x_j : j \in J]$ is its closed linear span.

In a Banach space X, a biorthogonal system

$$(x_j, u_j)_{j\in J}, \qquad x_j \in X, u_j \in X^*, \langle x_j, u_j\rangle = 1, \langle x_j, u_h\rangle = 0, j \neq h, j, h \in J,$$

is a Markushevich basis if $[x_j : j \in J]$ coincides with X and $\text{lin}\{u_j : j \in J\}$ is weak-star dense in X^*.

If S is a compact topological space, $C(S)$ is the real vector space of the continuous real-valued functions defined on S with the usual norm. S is said to be Corson if it is homeomorphic to a subspace T of the product $\mathbb{R}^J$, for some J depending on S, where $\mathbb{R}$ is the set of reals equipped with the usual topology, such that if the point $(a_j : j \in J)$ is in T, then the set $\{j \in J : a_j \neq 0\}$ is countable. S is an Eberlein compact if it is homeomorphic to a weakly compact subset of a Banach space. Using a result of Amir and Lindenstrauss [1], if S is Eberlein, then it is homeomorphic to a weakly compact subset A of the real space $c_0(J)$, for some index set J depending on S; clearly, the mapping φ from A to $\mathbb{R}^J$ which assigns to each element $(a_j : j \in J)$ in A the element $(a_j : j \in J)$ of $\mathbb{R}^J$ is a homeomorphism from A onto $\varphi(A)$ with $\{j \in J : a_j \neq 0\}$ countable, hence S is a Corson compact.

We have shown two results, in [4] and [5], respectively, that are more general than the following: (a) *If X is an Asplund space admitting a Markushevich basis $(x_j, u_j)_{j\in J}$ such that, for each u in X^*, the set $\{j \in J : \langle x_j, u\rangle \neq 0\}$ is countable, then X is weakly compactly generated.* (b) *If S is a Corson compact and E is a subspace of $C(S)$ that is closed for the topology of pointwise convergence, there is a Markushevich basis $(f_j, u_j)_{j\in J}$ for E such that, for each s in S, the set $\{j \in J : f_j(s) \neq 0\}$ is countable.*

Let S and T be two topological spaces. Let φ be a set-valued map from S to T. A mapping ψ from S to T is said to be a selector of φ whenever $\psi(s)$ belongs to $\varphi(s)$ for each s in S. The mapping φ is upper semicontinuous provided there is, for each s in S and each open neighborhood V of $\varphi(s)$, a neighborhood U of s such that $\varphi(x)$ is contained in V for every x in U.

We shall need the following result [3]: (c) *Let S be a metric space and let Y be a Banach space such that Y^* has the Radon-Nikodym property. Let φ be a set-valued map from S to Y^*. If φ is upper semicontinuous and $\varphi(x)$ is a non-empty compact set for each x in X respect to the weak-star topology of Y^*, then there is a selector ψ of φ of the first Baire class between the metric spaces S and Y^*.*

We say that a mapping f between the topological spaces S and T is *quasi-Baire* if there is a countable set L of continuous mappings from S to T such that f belongs to the closure of L in the topological space T^S.

The following result that we have shown in [6] will also be needed: (d) *Let X be a real Banach space. If there is a quasi-Baire mapping Φ from X to X^**

such that $\|\Phi(x)\| = 1$, $\langle x, \Phi(x)\rangle = \|x\|$, $x \in X$, $x \neq 0$, *then* X *is an Asplund space.*

Theorem 1. *If the Banach spaces* $X_1, X_2, \ldots, X_m$ *are all Asplund, then* $\mathscr{M}_{w^*}(X_1^*, X_2^*, \ldots, X_m^*)$ *is also Asplund.*

Proof. In order to simplify notation, let us write

$$Y := \mathscr{M}_{w^*}(X_1^*, X_2^*, \ldots, X_m^*)$$

and B instead of $B(X_1^*) \times B(X_2^*) \times \cdots \times B(X_m^*)$ provided with the topology induced by the weak-star topology of $X_1^* \times X_2^* \times \cdots \times X_m^*$.

For each f in Y, we define $\varphi(f)$ as the set of points $x \in B$ for which $f(x) = \|f\|$. It is plain that $\varphi(f)$ is non-empty and compact. Also, it is not hard to see that the set-valued map φ from Y to $X_1^* \times X_2^* \times \cdots \times X_m^*$ is upper semicontinuous when the latter space has the weak-star topology. Result (c) yields then a selector ψ of φ of the first Baire class between the metric spaces Y and $X_1^* \times X_2^* \times \cdots \times X_m^*$. Hence, there is a sequence (ψ_n) of continuous mappings from Y to $X_1^* \times X_2^* \times \cdots \times X_m^*$ such that, for each f in Y,

$$\lim_n \psi_n(f) = \psi(f).$$

For each f in Y, we put $f = f_1 + if_2$, with f_1 and f_2 real valued,

$$\|f_1\| := \sup\{|f_1(x_1, x_2, \ldots, x_m)| : (x_1, x_2, \ldots, x_m) \in B\}. \tag{1}$$

By setting $Z := \{f_1 : f \in Y\}$ we obtain a real vector space Z in which we consider the norm described in (1). If Y_r denotes the real Banach space subjacent to Y and we put $\Gamma f = f_1$, $f \in Y_r$, then

$$\Gamma : Y_r \to Z$$

is an onto linear map. Besides, if $(x_1, x_2, \ldots, x_m)$ belongs to B and f is in Y, we have

$$\begin{aligned} f(ix_1, x_2, \ldots, x_m) &= f_1(ix_1, x_2, \ldots, x_m) + if_2(ix_1, x_2, \ldots, x_m) \\ &= if_1(x_1, x_2, \ldots, x_m) - f_2(x_1, x_2, \ldots, x_m), \end{aligned}$$

and thus

$$\|f_1\| \leq \|f\| \leq 2\|f_1\|,$$

concluding that Γ is a topological isomorphism from Y_r onto Z. It all reduces to show that Z is an Asplund space, since then Y_r, and thereby Y, will also be Asplund.

Each element $(u_1, u_2, \ldots, u_m)$ of $X_1^* \times X_2^* \times \cdots \times X_m^*$ defines a continuous linear form $\Lambda(u_1, u_2, \ldots, u_m)$ on Z by putting, for each g of Z,

$$\langle g, \Lambda(u_1, u_2, \ldots, u_m)\rangle = g(u_1, u_2, \ldots, u_m).$$

We write

$$\Phi := \Lambda \circ \psi \circ \Gamma^{-1}, \qquad \Phi_n = \Lambda \circ \psi_n \circ \Gamma^{-1}, \quad n = 1, 2, \ldots.$$

The mappings $\Phi_n : Z \to Z^*$ are then continuous, $n = 1, 2, \ldots$, and

$$\lim_n \Phi_n(g) = \Phi(g), \qquad g \in Z.$$

Therefore,

$$\Phi : Z \to Z^*$$

is a quasi-Baire map. Let us write $M := \Lambda(B)$. Clearly, M is contained in $B(Z^*)$ and so, for each f in Y, $\|\Phi(f_1)\| \leq 1$. Moreover, if $(u_1, u_2, \dots, u_m)$ is the element of B such that $\psi(f) = (u_1, u_2, \dots, u_m)$, then

$$\begin{aligned}\|f_1\| \leq \|f\| &= f(u_1, u_2, \dots, u_m)\\ &= \langle f_1, \Phi(f_1)\rangle \leq \|f_1\| \cdot \|\Phi(f_1)\|,\end{aligned}$$

and we have

$$\|\Phi(f_1)\| = 1, \quad \langle f_1, \Phi(f_1)\rangle = \|f_1\|, \qquad f_1 \in Z, f_1 \neq 0.$$

We apply now result (d) to conclude that Z is Asplund. Q.E.D.

Corollary 1.1. *Let m be a positive integer. If X is an Asplund space, then $\mathscr{P}_{w^*}({}^mX^*)$ is an Asplund space.*

Proof. In the previous theorem we take $X = X_1 = X_2 = \cdots = X_m$. Then subspace F of $\mathscr{M}_{w^*}(X_1^*, X_2^*, \dots, X_m^*)$ consisting of the symmetric m-linear forms is a Banach space which is Asplund. We conclude by recalling that F is isomorphic to $\mathscr{P}_{w^*}({}^mX^*)$. Q.E.D.

The following result can be found in [2]: (e) *Let X be a Banach space with no copy of l^1. Let A be a weak-star compact subset of X^*. If B is the closed absolutely convex hull of A in X^*, then B is weak-star compact.*

Theorem 2. *Let X be a Banach space. If, for a positive integer m, $\mathscr{P}_{w^*}({}^mX^*)$ does not contain a copy of l^1, then $\mathscr{P}_{w^*}({}^mX^*)^{**}$ identifies with $\mathscr{P}_{(w^*)}({}^mX^*)$.*

Proof. Let λ be the map from X^* to $\mathscr{P}_{w^*}({}^mX^*)^*$ such that, for each u in X^* and each g in $\mathscr{P}_{w^*}({}^mX^*)$, $\langle\lambda(u), g\rangle = g(u)$. We set $A := \lambda(B(X^*))$. It can be simply checked that A is a weak-star compact subset of $\mathscr{P}_{w^*}({}^mX^*)^*$. Now, let B stand for the closed absolutely convex hull of A in the Banach space $\mathscr{P}_{w^*}({}^mX^*)^*$. In light of the previously mentioned result (e), we have that B is also weak-star compact. We deduce from this that, if v is a non-zero weak-star continuous linear form defined on $\mathscr{P}_{w^*}({}^mX^*)^*$, then $v \circ \lambda$ is a non-zero element of $\mathscr{P}_{w^*}({}^mX^*)$. Consequently, B is the closed unit ball of $\mathscr{P}_{w^*}({}^mX^*)^*$.

Let Λ be a map from X^* to $\mathscr{P}_{(w^*)}({}^mX^*)^*$ such that for each u in X^* and each g in $\mathscr{P}_{(w^*)}({}^mX^*)$,

$$\langle g, \Lambda u\rangle = g(u).$$

In $\mathscr{P}_{(w^*)}({}^mX^*)^*$, let H be the linear hull of $\Lambda(X^*)$ and E the closure of H. E is a Banach space. By Γ we denote the mapping from E to $\mathscr{P}_{w^*}({}^mX^*)^*$ which assigns to each u in E its restriction to $\mathscr{P}_{w^*}({}^mX^*)$. We next show that Γ is an isometry.

Obviously, $\lambda = \Gamma \circ \Lambda$. If g belongs to $\mathscr{P}({}^mX^*)$ and $\alpha \in \mathbb{C}$, then $g(\alpha x) = \alpha^m g(x)$, and thus, if, for a positive integer p, $B_p(X^*)$ represents the closed unit ball in X^* of radius 2^{-p}, the closed absolutely convex hull of $\Lambda(B_p(X^*))$ in $\mathscr{P}_{(w^*)}({}^mX^*)^*$ coincides with the closed ball in E of radius 2^{-pm}. Similarly, the closed absolutely convex hull of $\lambda(B_p(X^*))$ in $\mathscr{P}_{w^*}({}^mX^*)^*$ is the closed ball in $\mathscr{P}_{w^*}({}^mX^*)^*$ of radius 2^{-pm}.

Now, let P and Q be the absolutely convex hulls of $\Lambda(B(X^*))$ and $\lambda(B(X^*))$ in E and $\mathscr{P}_{w^*}({}^mX^*)^*$, respectively. Take a non-zero element w of E. We find $0 < \beta < 1$ such that $\|\beta w\| < 2^{-2m}$. We choose w_1 in P so that

$$\|\beta w - 2^{-2m} w_1\| < 2^{-4m}.$$

Proceeding by induction, let us assume that, for a positive integer p, we have found $w_1, w_2, \dots, w_p$ in P such that

$$\|\beta w - 2^{-2m} w_1 - 2^{-4m} w_2 - \cdots - 2^{-2pm} w_p\| < 2^{-2(p+1)m}.$$

We then determine w_{p+1} in P for which

$$\|2^{2(p+1)m}(\beta w - 2^{-2m} w_1 - 2^{-4m} w_2 - \cdots - 2^{-2pm} w_p) - w_{p+1}\| < 2^{-2m}$$

and

$$\|\beta w - 2^{-2m} w_1 - 2^{-4m} w_2 - \cdots - 2^{-2pm} w_p - 2^{2(p+1)m} w_{p+1}\| < 2^{-2(p+2)m}.$$

Hence, in E, we have

$$\beta w = \sum_{p=1}^{\infty} 2^{-2pm} w_p.$$

We take now a finite subset A_p of $B_p(X^*)$ such that $2^{-pm} w_p$ is in the absolutely convex hull in E of $\Lambda(A_p)$. If K is the closed absolutely convex hull in E of $\bigcup_{p=1}^{\infty} \Lambda(A_p)$ we have that K is compact and βw is clearly in K. Fix an element g of $\mathscr{P}_{(w^*)}({}^m X^*)$ such that $\langle g, w\rangle \neq 0$. We select an element f of $\mathscr{P}_{w^*}({}^m X^*)$ satisfying

$$|f(x) - g(x)| < \frac{1}{2}\beta|\langle g, w\rangle|, \qquad x \in \bigcup_{p=1}^{\infty} A_p.$$

Then

$$|\langle f - g, \Lambda(x)\rangle| = |f(x) - g(x)| < \frac{1}{2}\beta|\langle g, w\rangle|, \qquad x \in \bigcup_{p=1}^{\infty} A_p,$$

hence

$$|\langle f - g, \beta w\rangle| \leq \frac{1}{2}\beta|\langle g, w\rangle|,$$

and therefore

$$\begin{aligned} |\langle f, w\rangle| &\geq |\langle f, \beta w\rangle| \geq |\langle g, \beta w\rangle| - |\langle f - g, \beta w\rangle| \\ &\geq \beta|\langle g, w\rangle| - \frac{1}{2}\beta|\langle g, w\rangle| = \frac{1}{2}\beta|\langle g, w\rangle| \neq 0, \end{aligned}$$

thus proving Γ to be one-to-one.

Take now v in $\mathscr{P}_{w^*}({}^m X^*)^*$. We proceed as before and find $\gamma > 0$ and v_p in Q, $p = 1, 2, \dots,$ such that in $\mathscr{P}_{w^*}({}^m X^*)^*$

$$\gamma v = \sum_{p=1}^{\infty} 2^{-2pm} v_p.$$

It is then possible to determine a null sequence (x_q) in $B(X^*)$ and $\alpha_q > 0$, $q = 1, 2, \dots, \sum_{q=1}^{\infty} \alpha_q \leq 1$, so that

$$\gamma v = \sum_{q=1}^{\infty} \alpha_q \lambda(x_q).$$

But the series $\sum_{q=1}^{\infty} \alpha_q \Lambda(x_q)$ converges to an element γz in E. Thus

$$\Gamma(z) = \gamma^{-1} \sum_{q=1}^{\infty} \alpha_q (\Gamma \circ \Lambda)(x_q) = \gamma^{-1} \sum_{q=1}^{\infty} \alpha_q \lambda(x_q) = v\,,$$

and we have that Γ is onto.

Having in mind that $\Gamma(P) = Q$, Banach's isomorphism theorem guarantees that

$$\Gamma\colon E \to \mathscr{P}_{w^*}({}^m X^*)^*$$

is an isometry.

As a by-product of this we have that $B(\mathscr{P}_{w^*}({}^m X^*))$ is $\sigma(\mathscr{P}_{(w^*)}({}^m X^*), E)$-dense in $B(\mathscr{P}_{(w^*)}({}^m X^*))$, and since this set of linear forms on E is equicontinuous, it follows that in $B(\mathscr{P}_{(w^*)}({}^m X^*))$ the compact open topology of $\mathscr{P}({}^m X^*)$ coincides with $\sigma(\mathscr{P}_{(w^*)}({}^m X^*), E)$; thus $B(\mathscr{P}_{(w^*)}({}^m X^*))$ is compact for the topology of pointwise convergence on X^*. The desired result is now immediate. Q.E.D.

Corollary 1.2. *Let m be a positive integer. If the Banach space X is Asplund, then $\mathscr{P}_{(w^*)}({}^m X^*)$ is the bidual of $\mathscr{P}_{w^*}({}^m X^*)$.*

Proof. It follows from Corollary 1.1 that $\mathscr{P}_{w^*}({}^m X^*)$ is Asplund and so it does not contain copies of l^1. Our previous theorem now applies. Q.E.D.

Theorem 3. *Let X be a Banach space such that X^* has the approximation property. If, for a positive integer m, $\mathscr{P}_{w^*}({}^m X^*)$ does not contain copies of l^1, then $\mathscr{P}_{w^*}({}^m X^*)^{**}$ coincides with $\mathscr{P}({}^m X^*)$.*

Proof. For a positive integer p, we write $\mathscr{L}_s({}^p X^*)$ meaning the subspace of $\mathscr{M}(X^*, X^*, \dots, X^*)$ consisting of all symmetric p-linear forms. This space identifies in the usual fashion with the Banach space $\mathscr{L}(X^*, \mathscr{L}_s({}^{p-1} X^*))$ of the bounded linear operators from X^* to $\mathscr{L}_s({}^{p-1} X^*)$, assuming $\mathscr{L}_s({}^{p-1} X^*) = \mathbb{C}$ for $p = 1$.

Let f be in $\mathscr{P}({}^m X^*)$. We want to show that, given $\varepsilon > 0$ and a compact subset K of X^*, there is an element g in $\mathscr{P}_{w^*}({}^m X^*)$ such that $|f(x) - g(x)| < \varepsilon$, $x \in K$. To do this we follow a complete induction procedure. If $m = 1$, then f is an element of X^{**}, so there is an element g of X, hence of $\mathscr{P}_{w^*}({}^m X^*)$, such that $|f(x) - g(x)| < \varepsilon$, $x \in K$. We assume the property true for a positive integer $m - 1$ and show it still holds for m. We may assume, without loss of generality, that K is contained in $B(X^*)$. Let h be the element of $\mathscr{L}_s({}^m X^*)$ such that $h(x, x, \dots, x) = f(x)$, $x \in X^*$. For x_1 in X^*, $h(x_1, \cdot, \cdot, \dots, \cdot)$ is the element of $\mathscr{L}_s({}^{m-1} X^*)$ satisfying

$$h(x_1, \cdot, \cdot, \dots, \cdot)(x_2, x_3, \dots, x_m) = h(x_1, x_2, x_3, \dots, x_m).$$

We get hold of an operator

$$\zeta\colon X^* \to \mathscr{L}_s({}^{m-1} X^*)$$

of finite rank such that

$$\|\zeta(x) - h(x, \cdot, \cdot, \dots, \cdot)\| < \frac{\varepsilon}{2}, \qquad x \in K.$$

We find $h_1, h_2, \dots, h_q$ in $\mathscr{L}_s({}^{m-1} X^*)$ and $u_1, u_2, \dots, u_q$ in X^{**} so that

$$\zeta(x) = u_1(x) h_1 + u_2(x) h_2 + \cdots + u_q(x) h_q\,, \qquad x \in X^*.$$

We take now v_j in X, $j = 1, 2, \dots, q$, such that

$$|v_j(x) - u_j(x)| \cdot \|h_j\| < \frac{\varepsilon}{4q}, \qquad x \in K.$$

By the induction hypothesis, there is a polynomial κ_j in $\mathscr{P}_{w^*}({}^{m-1}X^*)$ for which

$$|h_j(x, x, \dots, x) - \kappa_j(x)| \cdot \|v_j\| < \frac{\varepsilon}{4q}, \qquad x \in K.$$

Then

$$g(x) := v_1(x)\kappa_1(x) + v_2(x)\kappa_2(x) + \cdots + v_q(x)\kappa_q(x), \qquad x \in X^*,$$

is an element of $\mathscr{P}_{w^*}({}^m X^*)$. For each x in K, we have

$$\begin{aligned}
&|\zeta(x)(x, x, \dots, x) - g(x)| \\
&\quad \le \sum_{j=1}^{g} |(u_j(x) - v_j(x))h_j(x, x, \dots, x)| \\
&\quad\quad + \sum_{j=1}^{q} |v_j(x)(h_j(x, x, \dots, x) - \kappa_j(x))| \\
&\quad \le \sum_{j=1}^{q} |u_j(x) - v_j(x)| \cdot \|h_j\| + \sum_{j=1}^{q} \|v_j\| \cdot |h_j(x, x, \dots, x) - \kappa_j(x)| \le \frac{\varepsilon}{2}.
\end{aligned}$$

Finally, for each x in K,

$$\begin{aligned}
|f(x) - g(x)| &= |h(x, x, \dots, x) - g(x)| \\
&\le |h(x, x, \dots, x) - \zeta(x)(x, x, \dots, x)| \\
&\quad + |\zeta(x)(x, x, \dots, x) - g(x)| \\
&\le \|h(x, \cdot, \dots, \cdot) - \zeta(x)\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \text{Q.E.D.}
\end{aligned}$$

Corollary 1.3. *Let X be an Asplund space such that X^* has the approximation property. If m is a positive integer, $\mathscr{P}_{w^*}({}^m X^*)^{**}$ coincides with $\mathscr{P}({}^m X^*)$.*

Proof. It follows immediately from Corollary 1.1 and the last theorem. Q.E.D.

Theorem 4. *Let $X_1, X_2, \dots, X_m$ be weakly compactly generated Asplund spaces. If G is a subspace of $\mathscr{M}_{w^*}(X_1^*, X_2^*, \dots, X_m^*)$, closed for the topology of pointwise convergence, then G is a weakly compactly generated Banach space.*

Proof. Our notation will be that used in the proof of Theorem 1. For $j = 1, 2, \dots, m$, let M_j be a weakly compact absolutely convex subset whose linear span is dense in X_j. If M_j^0 is the polar set of M_j in X_j^*, then this linear space, taking M_j^0 as closed unit ball, is a normed space in which $B(X_j^*)$ is weakly compact, hence $B(X_j^*)$ is a Corson compact for the weak-star topology. We have then that B is a Corson compact. It can be easily seen that Z is a closed subspace of $C(B)$ for the topology of pointwise convergence. We apply result (b) to obtain a Markushevich basis for $\Gamma(G)$ such that, for each x in B, the set $\{j \in J\colon g_j(x) \neq 0\}$ is countable. Following the proof of Theorem 1, we have that Z is Asplund and so it contains no copy of l^1. If D is the set formed by the restrictions of the elements of $\Lambda(B)$ to $\Gamma(G)$, we can apply result (e) to have that the closed absolutely convex hull of D in $\Gamma(G)^*$ is weak-star compact, and thus it coincides with the closed unit ball of this space. From this we have that, for each u in $\Gamma(G)^*$, $\{j \in J: \langle g_j, u\rangle \neq 0\}$ is countable. Result (a) assures then that $\Gamma(G)$ is weakly compactly generated. If G_r denotes the real Banach space subjacent to G, it follows that

$$\Gamma_{|G_r}: G_r \to \Gamma(G_r)$$

is an isomorphism, and thereby G_r is weakly compactly generated. It is now immediate that G is also weakly compactly generated. Q.E.D.

Corollary 1.4. *Let m be a positive integer. If X is a weakly compactly generated Asplund space, then $\mathscr{P}_{w^*}(^m X^*)$ is weakly compactly generated.*

Proof. The subspace F of $\mathscr{M}_{w^*}(X_1^*, X_2^*, \dots, X_m^*)$ formed by the symmetric m-linear forms when $X_1^* = X_2^* = \cdots = X_m^* = X$ is closed for the topology of pointwise convergence and so, in light of last theorem, F is weakly compactly generated. Since F is isomorphic to $\mathscr{P}_{w^*}(^m X^*)$, the conclusion follows. Q.E.D.

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