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A NOTE ON THE CONSTRUCTION OF A CERTAIN CLASS OF KLEINIAN GROUPS

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ABSTRACT. We prove that if $\{S_1, S_1', \dots, S_n, S_n'\}$ is a collection of distinct spheres in $\mathbb{R}^m$ with common exterior, and $g_1, \dots, g_n$ are Möbius transformations such that for each i, S_i is the isometric sphere of g_i and S_i' is the isometric sphere of g_i^{-1} and such that g_i maps points of contact of S_i to points of contact of S_i', then the group G generated by the g_i's is Kleinian.

1. INTRODUCTION

A practical way to construct examples of Kleinian groups (acting in the real hyperbolic space as a group of isometries or in $\overline{\mathbb{R}}^m = \mathbb{R}^m \cup \{\infty\}$ as a group of conformal transformations) is to choose spheres $\{S_1, S_1', \dots, S_n, S_n'\}$ in $\overline{\mathbb{R}}^m$ and Möbius transformations $g_1, \dots, g_n$ such that g_i maps the outside of S_i to the inside of S_i'. But this is not enough to guarantee that the group generated by $g_1, \dots, g_n$ is Kleinian.

In [1] M. Bestvina and D. Cooper intended to give an example of a Kleinian group whose limit set in S^3 is a wild Cantor set using this idea. But, as was pointed out in [2], there is a gap in their result because it is not verified there whether the group obtained is Kleinian. In section 2.5 we give an example due to B. Maskit which shows that this technique must be used carefully.

In this paper we show that if we take care of some symmetry conditions and a particular choice of the Möbius transformations the group obtained is Kleinian. Basically we show that in this case some parabolicity condition is satisfied, thus making unnecessary the (sometimes troublesome) algebraic verification.

In section 2 we present the basic definitions and Poincaré's Polyhedron Theorem (in a version sufficient to our purposes) and then define the particular setting for the construction of a Kleinian group. Then in section 3 we state precisely what needs to be proved and then sketch the proof of this fact. In

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section 4 we prove the main result. And in section 5 we discuss some future work that can be done in these lines.

2. Preliminaries

2.1. Let us first set the terminology. We will follow Maskit's book [3].

We can look at a Möbius transformation either as an isometry of the hyperbolic space $\mathbb{H}^{m+1}$ or as a conformal transformation of its boundary at infinity, which can be modelled by $\overline{\mathbb{R}}^m = \mathbb{R}^m \cup \{\infty\}$.

If a Möbius transformation g does not fix ∞, then there is a Euclidean sphere S centered at $g^{-1}(\infty)$ which is mapped by g to a sphere S' of the same radius centered at $g(\infty)$. The sphere S is called the *isometric sphere* of g. It can be proved that g can be decomposed as $g = q \circ p$ where p is the inversion in S and q is a Euclidean motion (see [3, Chapter 4, section G]).

A *Kleinian group* is a group of Möbius transformations which acts properly discontinuously at some open set in $\mathbb{H}^{m+1}$ or in $\overline{\mathbb{R}}^m$.

A (finite) *polyhedron* F' in $\mathbb{H}^{m+1}$ is a nonempty open region bounded by finitely many hyperbolic hyperplanes (which are called the *sides* of F'). Their boundaries or *traces* in $\overline{\mathbb{R}}^m$ are either Euclidean spheres or hyperplanes (i.e., spheres passing through ∞). So these define a region F in $\overline{\mathbb{R}}^m$ which is the trace of F'. An *edge* is the intersection of two sides of F'. An *edge at infinity* is the tangency point of the trace of two sides which do not intersect but have distance zero. We define the angle $\alpha(e)$ of an edge e as the angle between the sides defining it measured from the inside of F'.

2.2. Assume that for each side s of F' there is a given Möbius transformation g_s such that there is a side s' of F' such that $g_s(s) = s'$. These g_s are called *side pairing transformations* of F'.

Observe that edges are mapped to edges by the side pairing transformations so they come in cycles under their action. The same happens to the edges at infinity. Let $\{e_1, \ldots, e_k\}$ be a cycle of edges (respectively edges at infinity) and let $\{g_1, \ldots, g_k\}$ be side pairing transformations such that $g_i(e_i) = e_{i+1}$ if $1 \le i < k$ and $g_k(e_k) = e_1$. Let $h_e = g_k \circ \cdots \circ g_1$. This is called *cycle transformation* of the edge $e = e_1$ (respectively *infinite cycle transformation* of the edge at infinity e).

Let G be the group generated by the g_s (for all sides s of F').

2.3. Consider the following conditions:

1. $g_s(s) = s'$;
2. $g_{s'} = g_s^{-1}$;
3. $g_s(F') \cap F' = \varnothing$;
4. the orbit under G of any $x \in \overline{F}'$ has only finitely many points in $\overline{F}'$;
5. for each edge e there is a positive integer t such that $h_e^t = 1$;
6. for each cycle of edges $\{e_1, \ldots, e_k\}$ let t be the minimal i such that $h_e^i = 1$, $e = e_1$; then $\sum_{j=1}^k \alpha(e_j) = 2\pi/t$;
7. the infinite cycle transformation at an infinite edge is parabolic.

Then

Theorem 1 (Poincaré's Polyhedron Theorem). *If the conditions* (1) *through* (7) *above are satisfied, then the group* G *is Kleinian and* F' *is a fundamental domain for* G.

By [3, Chapter 6, section A.3], the trace F of F' in $\overline{\mathbb{R}}^m$ is a fundamental domain for G acting on $\overline{\mathbb{R}}^m$ as a group of conformal transformations. Of course we can extend the action of a group of conformal transformations of $\overline{\mathbb{R}}^m$ to $\mathbb{H}^{m+1}$. If this extended action gives a Kleinian group, then the original one also gives a Kleinian group.

So, consider the following setting.

2.4. Let $\mathscr{B} = \{B_1, B_1', \dots, B_n, B_n'\}$ be a collection of closed metric balls in $\mathbb{R}^m$, such that:

1. the radius of B_i equals the radius of B_i', for all i;
2. each ball lies outside the others, that is, if $i \neq j$, $\operatorname{int}(B_i) \cap \operatorname{int}(B_j) = \operatorname{int}(B_i') \cap \operatorname{int}(B_j') = \varnothing$, and for all k, l, $\operatorname{int}(B_k) \cap \operatorname{int}(B_l') = \varnothing$;
3. they can touch.

Put $S_i = \partial B_i$ and $S_i' = \partial B_i'$.

Let $g_1, \dots, g_n$ be (orientation-preserving) Möbius transformations such that:

- for each i, S_i is the isometric sphere of g_i and S_i' is the isometric sphere of g_i^{-1};
- g_i maps points of contact of S_i to points of contact of S_i'.

We are going to prove that the group G generated by the g_i's is a Kleinian group. We observe that the conditions (1) to (6) are automatically satisfied. Conditions (5) and (6) are satisfied because there are no edges other than the infinite ones in this setting. We need only to verify if the infinite cycle transformations at every infinite edge are parabolic.

2.5. **An example**. We call attention to the following example due to Maskit [4], which shows that this verification is really necessary.

Let $S_1 = \{z \in \mathbb{C}\colon |z| = 1\}$, $S_1' = \{z \in \mathbb{C}\colon |z| = 3\}$, $S_2 = \{z \in \mathbb{C}\colon |z+2| = 1\}$ and $S_2' = \{z \in \mathbb{C}\colon |z-2| = 1\}$, and $g_1(z) = 3z$, $g_2(z) = (2z+5)/(z+2)$. It is easy to verify that S_2 is the isometric circle of g_2 and S_2' the isometric circle of g_2^{-1}. But we cannot speak of isometric circles of g_1 and g_1^{-1} because they fix ∞. Let D be the region bounded by these spheres, i.e. between S_1 and S_1' and outside S_2 and S_2'. This setting satisfies conditions (1) through (6) but not the parabolicity condition (7). (The infinite cycle transformation of the contact point P of S_1 and S_2 is $h = g_1^{-1} g_2^{-1} g_1^{-1} g_2$; the reader can verify that h is loxodromic, not parabolic.) In fact, Maskit shows that D is not a fundamental domain for the group generated by g_1 and g_2.

So we stress the importance of the hypothesis that the spheres S_i and S_i' be the isometric spheres of g_i and g_i' respectively, in our result.

3. Statement of the result and sketch of the proof

3.1. Let $h = a_k a_{k-1} \dots a_1$ be an infinite cycle transformation at an infinite edge (or contact point), where $a_i \in \{g_1, \dots, g_n\} \cup \{g_1^{-1}, \dots, g_n^{-1}\}$. Let I_j and I_j' be the isometric spheres of a_j and a_j^{-1} respectively. We have that each a_j can be decomposed as $a_j = q_j p_j$ where p_j is the inversion in I_j and q_j is an euclidean motion. (See [3, Chapter 4, section G].) For each $j = 1, \dots, k$, let R_j, R_j', T_j, T_j' denote the open regions inside I_j and I_j' and outside I_j and I_j' respectively.

Thus h is a cycle transformation of the contact point X_1 of I_1 and I'_k. We want to show that

Theorem 2. *X_1 is the only fixed point of h in* $\mathbb{R}^n \cup \{\infty\}$ (and hence h is parabolic).

3.2. We first show that there is no fixed point of h in $T_1 \cup T'_k \cup \{\infty\}$ (the region outside the isometric spheres I_1 and I'_k) by showing that the image under h of $T_1 \cup T'_k \cup \{\infty\}$ lies inside R'_k, which is disjoint of this region. Next we show that there is no fixed point of h in R'_k (the region inside I'_k). This we do by showing that the image under h of any sphere $S \subseteq \overline{T}_1$ (i.e., not inside I_1), tangent to I_1 at X_1, has diameter strictly less than the diameter of S. The same argument then applies to show that there is no fixed point of h^{-1} in R_1 (the region inside I_1). Since the fixed points of h^{-1} are exactly those of h, this proves that the only fixed point of h is X_1. Hence h is parabolic.

4. The proof

We divide the proof in the following lemmas proving the three steps sketched above. Recall the decomposition $h = a_k a_{k-1} \cdots a_1$, with $a_i \in \{g_1, \dots, g_n\} \cup \{g_1^{-1}, \dots, g_n^{-1}\}$. **Warning**: All the metric notions used here (e.g., sphere, its center, its diameter, etc.) are Euclidean.

4.1. **Lemma.** *There is no fixed point of h in $T_1 \cup T'_k \cup \{\infty\}$ (the region outside the isometric spheres I_1 and I'_k).*

Proof. Let $U_0 = T_1 \cup T'_k \cup \{\infty\}$. Then inversion in I_1 maps U_0 into R_1. So a_1 maps U_0 into R'_1 (because of the decomposition of $a_1 = q_1 p_1$, with p_1 the inversion in I_1 and q_1 a Euclidean motion which maps R_1 into R'_1). But R'_1 lies outside I_2, hence a_2 maps $U_1 = a_1(U_0)$ into R'_2. Inductively we have that if $U_i = a_i(U_{i-1})$, then $U_i \subseteq R'_i$. But $U_k = h(U_0)$. This proves the lemma. □

4.2. **Lemma.** *If A_1 is a sphere of diameter $d_1 > 0$ tangent to I_1 at X_1 and lying not inside I_1, then the diameter of $h(A_1)$ is strictly less than d. (Observe that $h(A_1)$ is a sphere lying not inside I_1 and tangent to I_1 at X_1; see Figure 1.)*

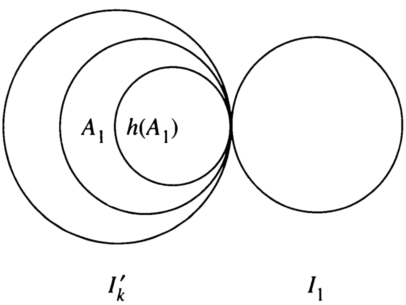

Figure 1. Statement of Lemma 4.2

Proof. We do this inductively by showing that the image under a_i of a sphere A_i of diameter $d_i > 0$ tangent to I_1 at $X_i = a_{i-1}(X_{i-1})$ (this is the point of contact of I'_{i-1} and I_i) is a sphere of diameter $d_{i+1} < d_i$ tangent to I_{i+1}

at $X_{i+1} = a_i(X_i)$ (and making $I_{k+1} = I_1$, this proves that the diameter d_k of $h(B)$ is $d_k < d_1$).

So let r_i be the radius of I_i. Then the diameter of $p_i(A_i)$ (the inversion of A_i in I_i) is $d_{i+1} = r_i - r_i^2/(r_i + d_i)$. We have that:

$$d_i - d_{i+1} = d_i - \left(r_i - \frac{r_i^2}{r_i + d_i}\right) = \frac{d_i^2}{r_i + d_i} > 0,$$

i.e., $d_{i+1} < d_i$. (See Figure 2.)

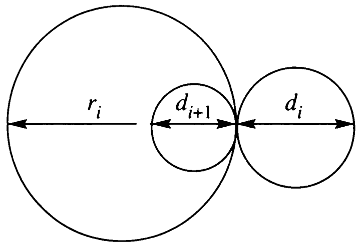

FIGURE 2. Comparison of diameters in the proof of Lemma 4.2

But $a_i(A_i) = q_i p_i(A_i)$ and q_i is a Euclidean motion mapping R_i into R_i'. This finishes the proof of the lemma. □

4.3. **Lemma**. *There is no fixed point of h in $I_k' \cup R_k'$ apart from X_1. (Observe that $X_1 \in I_k' \cap I_1$.)*

Proof. Let $X \in I_k' \cup R_k'$, $X \neq X_1$. Let A_1 be the sphere containing X and tangent to I_1 at X_1, and d_1 its radius. We observe that this sphere is unique. By Lemma 4.2 above, the radius of $h(A_1)$ is strictly less than d_1. Hence X cannot be in $h(A_1)$. (Actually, the only common point of A_1 and $h(A_1)$ is X_1, the point of tangency.) Therefore $h(X) \neq X$. □

4.4. **Lemma**. *There is no fixed point of h in $I_1 \cup R_1$ apart from X_1.*

Proof. The same proof of the previous lemma applies to prove that there is no fixed point of h^{-1} in $I_1 \cup R_1$. Since the fixed points of h^{-1} are exactly the same as the ones of h, this proves the lemma. □

With this we have finished the proof of Theorem 2.

5. CONCLUSION

By our result if we want to construct an example of a Kleinian group by taking touching spheres we need only to take care of using isometric spheres and to be sure that contact points go to contact points.

5.1. We believe that this construction can be useful to study deformations of this class of groups by simply describing their fundamental regions in terms of isometric spheres.

We ask the following questions.

Question 1. How does a deformation of G correspond to a modification of F?

Question 2. Can one reduce the problem of finding the dimension of the space of deformations of G to a problem of linear algebra? This means, can one find an algebraic or combinatorial invariant related to this space of deformations?

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