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ASYMPTOTICS OF REPRODUCING KERNELS ON A PLANE DOMAIN

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ABSTRACT. Let Ω be a plane domain of hyperbolic type, $|dz|/w(z)$ the Poincaré metric on Ω, and $K_{\Omega,q}(x,\overline{y})$ the reproducing kernel for the Hilbert space $\mathscr{A}_q^2(\Omega)$ of all holomorphic functions on Ω square-integrable with respect to the measure $w(z)^{2q-2}\,|dz\wedge d\overline{z}|$. It is proved that

$$\lim_{q\to+\infty}\frac{K_{\Omega,q}(z,\overline{z})w(z)^{2q}}{2q}=\frac{1}{\pi}.$$

Let $\Omega\subset\mathbf{C}$ be a domain of hyperbolic type (i.e., $\mathbf{C}\setminus\Omega$ contains at least two points), so that the universal covering surface of Ω is isomorphic to the unit disc $\mathbf{D}$. Let $\phi:\mathbf{D}\to\Omega$ be the covering map and

$$G=\{\omega\in\operatorname{Aut}(\mathbf{D}):\ \phi\circ\omega=\phi\}$$

the corresponding group of covering transformations. G acts freely and properly discontinuously on $\mathbf{D}$, and Ω may be identified with the coset space $\mathbf{D}/G$. The Poincaré metric on Ω, corresponding to the hyperbolic metric $\frac{|dz|}{1-|z|^2}$ on $\mathbf{D}$, is given by $ds=\frac{|dx|}{w(x)}$, where (see [1, section II.1])

$$w(\phi(z))=(1-|z|^2)\,|\phi'(z)|.$$

The identity

$$1-|\omega(z)|^2=(1-|z|^2)\,|\omega'(z)|,\qquad z\in\mathbf{D},\ \omega\in\operatorname{Aut}(\mathbf{D}),$$

which can be verified by a short computation, shows that the right-hand side indeed depends only on $\phi(z)$, so that the definition of $w(x)$ is correct.

The *Bergman space* $A^2_\alpha(\Omega)$ consists, by definition, of all holomorphic functions on Ω square-integrable against the measure $w(x)^\alpha\,dE(x)$, where dE is the Lebesgue area measure. We shall only be interested in the case when α is an even integer: $\alpha=2q-2$ ($q\in\mathbf{Z}$), and we will write $\mathscr{A}_q^2(\Omega)$ instead of $A^2_\alpha(\Omega)$ in that case. Endowed with the *Petersson scalar product*

$$\langle f,g\rangle_q=\int_\Omega f(x)\,\overline{g(x)}\,w(x)^{2q-2}\,dE(x),$$

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$\mathscr{A}_q^2(\Omega)$ becomes a Hilbert space. For $q \geq 2$ (or, if $\Omega \notin O_G$, for $q \geq 1$), it can be shown that $\mathscr{A}_q^2(\Omega) \neq \{0\}$ and that $\mathscr{A}_q^2(\Omega)$ admits a *reproducing kernel* (see [1, Chapter III]): there exists a function $K_{\Omega,q}(x, \overline{y})$ $(x, y \in \Omega)$, holomorphic in x and antiholomorphic in y, such that

$$f(x) = \langle f, K_{\Omega,q}(\cdot, \overline{x})\rangle_q = \int_\Omega f(y) K_{\Omega,q}(x, \overline{y})\, w(x)^{2q-2}\, dE(x),$$
$$\forall f \in \mathscr{A}_q^2(\Omega),\ x \in \Omega.$$

Our main goal is to prove the following result.

Theorem. *For any* $x \in \Omega$,

$$\lim_{q\to+\infty} \frac{K_{\Omega,q}(x, \overline{x})\, w(x)^{2q}}{2q} = \frac{1}{\pi}.$$

For $\Omega = \mathbf{D}$, this is trivial, and for Ω an annulus or a punctured disc, this has been obtained (by a totally different method) in [3] and applied to the study of the asymptotics of the Berezin transform. The latter, in turn, provides the basic tool for certain quantization procedures on Ω (construction of $*$-products) [4], [5].

Proof. We will borrow the notation and some results from Chapter III of Kra [1]. Let $\Delta \subset \mathbf{D}$ be a fundamental domain for G (see Tsuji [2, section XI.2]). Denote

$$\mathscr{O}(\mathbf{D}) = \{\text{holomorphic functions on } \mathbf{D}\},$$

$$\|f\|_{q,G} = \left(\int_\Delta |f(z)|^2 (1-|z|^2)^{2q-2}\, dE(z)\right)^{1/2},$$

$$\mathscr{A}_q^2(\mathbf{D}, G) = \{f \in \mathscr{O}(\mathbf{D}) :\ \|f\|_{q,G} < +\infty$$
$$\text{and } f(z) = f(\omega(z)) \cdot \omega'(z)^q \quad \forall \omega \in G\}.$$

The mapping

$$f(x) \mapsto f(\phi(z))\, \phi'(z)^q, \qquad x \in \Omega,\ z \in \mathbf{D},$$

establishes a Hilbert space isomorphism of $\mathscr{A}_q^2(\Omega)$ onto $\mathscr{A}_q^2(\mathbf{D}, G)$ [1, section III.6]. It follows that the function $F_{q,G}(z, \overline{y})$, defined by

$$F_{q,G}(z, \overline{y}) = K_{\Omega,q}(\phi(z), \overline{\phi(y)}) \cdot \phi'(z)^q \overline{\phi'(y)^q}, \qquad z, y \in \mathbf{D},$$

is the reproducing kernel for the Hilbert space $\mathscr{A}_q^2(\mathbf{D}, G)$. But, according to section III.5 of [1], the latter reproducing kernel is also given by the Poincaré series

$$F_{q,G}(z, \overline{y}) = \sum_{\omega \in G} K_{\mathbf{D},q}(\omega(z), \overline{y})\, \omega'(z)^q, \qquad z, y \in \mathbf{D},$$

where $K_{\mathbf{D},q}$ is the reproducing kernel for the Bergman space $\mathscr{A}_q^2(\mathbf{D})$ on the unit disc $\mathbf{D}$. Summarizing, we have

$$\lim_{q\to+\infty} \frac{K_{\Omega,q}(x, \overline{x})\, w(x)^{2q}}{2q} = \lim_{q\to+\infty} \frac{F_{q,G}(z, \overline{z}) |\phi'(z)|^{-2q} \cdot (1-|z|^2)^{2q} |\phi'(z)|^{2q}}{2q}$$
$$= \lim_{q\to+\infty} \frac{(1-|z|^2)^{2q}}{2q} \sum_{\omega \in G} K_{\mathbf{D},q}(\omega(z), \overline{z})\, \omega'(z)^q,$$

where $x = \phi(z) \quad (x \in \Omega,\ z \in \mathbf{D})$. Now it is well known that

$$K_{\mathbf{D},q}(y, \bar{z}) = \frac{2q-1}{\pi}(1 - \bar{z}y)^{-2q}.$$

Therefore

$$\lim_{q\to+\infty} \frac{(1-|z|^2)^{2q}}{2q} \sum_{\omega\in G} K_{\mathbf{D},q}(\omega(z), \bar{z})\,\omega'(z)^q$$
$$= \frac{1}{\pi} \lim_{q\to+\infty} \sum_{\omega\in G} \left[(1-|z|^2)^2(1-\bar{z}\omega(z))^{-2}\omega'(z)\right]^q.$$

For $a \in \mathbf{D}$, let $\omega_a \in \operatorname{Aut}(\mathbf{D})$ be the Möbius transformation given by

$$\omega_a(y) = \frac{y-a}{1-\bar{a}y}.$$

Then

$$\omega_a{}'(y) = \frac{1-|a|^2}{(1-\bar{a}y)^2}, \qquad \omega_a{}'(0) = 1 - |a|^2.$$

It follows that

$$\begin{aligned}(1-|z|^2)^2(1-\bar{z}\omega(z))^{-2}\omega'(z) &= (1-|z|^2)\cdot\omega_z{}'(\omega(z))\cdot\omega'(z)\\ &= \omega_{-z}{}'(0)\cdot\omega_z{}'(\omega(z))\cdot\omega'(z)\\ &= (\omega_z\omega\omega_{-z})'(0).\end{aligned}$$

Let $\lambda = \omega_z\omega\omega_{-z} \in \operatorname{Aut}(\mathbf{D})$. By the Schwarz lemma (or direct computation),

$$|\lambda'(0)| = 1 - |\lambda(0)|^2 \le 1,$$

with equality occurring iff $\lambda(0) = 0$, i.e. iff $\omega(z) = z$. Since G acts freely, this is only possible for $\omega = \mathrm{id}$. Thus

$$|(\omega_z\omega\omega_{-z})'(0)| < 1 \qquad \text{for } \omega \in G \setminus \{\mathrm{id}\}.$$

Let us now make the following elementary observation: *whenever* $\{b_\nu\}_{\nu\in I}$ *is a sequence of complex numbers, indexed by a countable set* I, *such that*

(a) $\sum_{\nu\in I} |b_\nu|^q < +\infty$ *for some* $q > 0$, *and*
(b) $|b_\nu| < 1$ *for all* ν,

then $\sum_{\nu\in I} b_\nu^q \to 0$ *as* $q \to +\infty$.

Indeed, it follows from (a) that for any $\epsilon > 0$ there is at most a finite number of b_ν with $|b_\nu| > \epsilon$; so there exists $\nu_0 \in I$ such that $|b_{\nu_0}| = \sup_I |b_\nu|$. Then

$$\left|\sum_I b_\nu^q\right| \le |b_{\nu_0}^q| \sum_I |b_\nu / b_{\nu_0}|^q,$$

and as $|b_\nu/b_{\nu_0}| \le 1$ for all ν, the last sum is a nonincreasing function of q. It follows that $\sum_I b_\nu^q = O(|b_{\nu_0}|^q)$, and as $|b_{\nu_0}| < 1$ by (b), the assertion follows.

Since

$$\sum_{\omega\in G} |(\omega_z\omega\omega_{-z})'(0)|^2 = \sum_{\omega\in G}(1-|\omega_z\omega(z)|^2)^2 = \sum_{\omega\in G}\left[\frac{(1-|z|^2)(1-|\omega(z)|^2)}{|1-\bar{z}\omega(z)|^2}\right]^2$$
$$\le \left(\frac{1+|z|}{1-|z|}\right)^2 \sum_{\omega\in G}(1-|\omega(z)|^2)^2$$

is finite by [1, Lemma III.5.2], the above observation can be applied to $I = G \setminus \{\mathrm{id}\}$, $b_\omega = (\omega_z \omega \omega_{-z})'(0)$, and shows that

$$\sum_{\omega \in G \setminus \{\mathrm{id}\}} [(\omega_z \omega \omega_{-z})'(0)]^q \to 0 \qquad \text{as } q \to +\infty.$$

Consequently,

$$\begin{aligned}\lim_{q \to +\infty} \sum_{\omega \in G} \left[(1-|z|^2)^2 (1-\overline{z}\omega(z))^{-2} \omega'(z)\right]^q &= \lim_{q \to +\infty} \sum_{\omega \in G} [(\omega_z \omega \omega_{-z})'(0)]^q \\ &= \lim_{q \to +\infty} [(\omega_z \cdot \mathrm{id} \cdot \omega_{-z})'(0)]^q \\ &= 1\end{aligned}$$

and the proof is finished.

Problem. In view of the result just proved, the following question now seems to be of some interest. Let F be a positive continuous function on the interval $[0, 1)$ and let K_α be the reproducing kernel for the Bergman space $A^2(\mathbf{D}, F(|z|^2)^\alpha \, dE(z))$. For which F do the functions

$$K_\alpha(z, \overline{z}) \, F(|z|^2)^\alpha / \alpha$$

converge as $\alpha \to +\infty$, and if they do, what is the limit?

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