A CHARACTERIZATION OF CLIFFORD MINIMAL HYPERSURFACES IN $S^4$

LI HAIZHONG

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ABSTRACT. In this note we give a characterization of Clifford minimal hypersurfaces in $S^4$ by the Ricci curvature condition.

1. Introduction and theorem

Let $S^4$ be a 4-dimensional unit sphere space. It is well known (see Chern, do Carmo, and Kobayashi [1]) that the closed Clifford hypersurface $C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$ in $S^4$ has two different principal curvatures $\lambda_1 = \sqrt{2}$, $\lambda_2 = \lambda_3 = -\sqrt{1/2}$. From the Gauss equation we easily know that the Ricci curvature of $C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$ satisfies

(1) $0 \leq \text{Ric}(C_{1,2}) \leq \frac{3}{2}$.

A natural problem is that if $M$ is a closed minimal hypersurface in $S^4$ and the Ricci curvature of $M$ satisfies $0 \leq \text{Ric}(M) \leq 3/2$, then is it the Clifford minimal hypersurface $C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$?

In this note we give an affirmative answer for the above problem, that is, we prove the following

**Theorem.** Let $M$ be a closed minimal hypersurface in $S^4$ which satisfies the Ricci curvature condition

(2) $0 \leq \text{Ric}(M) \leq \frac{3}{2}$.

Then $M = C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$.

2. Preliminaries

Let $M$ be an $n$-dimensional closed minimal hypersurface in an $(n + 1)$-dimensional unit sphere space $S^{n+1}$, and $e_1, \ldots, e_n$ a local orthonormal frame field on $M$, $\omega_1, \ldots, \omega_n$ its dual coframe field. Here for the sake of simplicity, we keep the notation of Peng and Terng [3]. We have the following formulas
(see [3]):

\[ I = \sum_i \omega_i^2, \]

\[ II = h = \sum_{i,j} h_{ij} \omega_i \omega_j, \quad h_{ij} = h_{ji}, \]

\[ \nabla h = \sum_{i,j,k} h_{ijk} \omega_i \omega_j \omega_k, \quad h_{ijk} = h_{ikj}, \]

\[ \nabla^2 h = \sum_{i,j,k,l} h_{ijkl} \omega_i \omega_j \omega_k \omega_l, \]

where

\[ h_{ijkl} = h_{ijlk} + \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}, \]

\[ \Delta h_{ij} = (n - S) h_{ij}, \quad \frac{1}{2} \Delta S = |\nabla h|^2 + S(n - S), \]

where \( S = |h|^2 \).

We introduce the following symmetric functions:

\[ f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}, \]

\[ f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}. \]

By using (7) and (8), we can compute the Laplacian of the function \( f_4 \) as follows:

\[ \Delta f_4 = 4[(n - S) f_4 + 2A + B], \]

where

\[ A = \sum_{i,j,k,l,m} h_{ijkl} h_{ijkl} h_{kml}, \quad B = \sum_{i,j,k,l,m} h_{ijkl} h_{klim} h_{ljm}. \]

Near any given point \( p \in M \), we can choose a local frame field \( e_1, \ldots, e_n \) so that at \( p \), we have

\[ h_{ij} = \lambda_i \delta_{ij}. \]

Then, from the symmetricity of \( h_{ijk} \) about indices, we have

\[ A + 2B = \frac{1}{3} \sum_{i,j,k} h_{ijk}^2 (\lambda_i + \lambda_j + \lambda_k)^2 \geq 0. \]

We need the following lemma to prove our theorem

**Lemma 1** (see Theorem 4 of Peng and Terng [3]). Let \( M \) be a closed minimal hypersurface in \( S^{n+1} \). Then

\[ \int_M S f_4 - f_3^2 - S^2 - \frac{|\nabla S|^2}{4} = \int_M A - 2B, \]

where \( A \) and \( B \) are defined by (12).
Integrating both sides of (11), we have

\[(15) \quad \int_M (S - n)f_4 = \int_M 2A + B.\]

By (13), \(-3 \times (14) + 4 \times (15)\) implies the following lemma

**Lemma 2.** Let \(M\) be a closed minimal hypersurface in \(S^{n+1}\). Then

\[(16) \quad \int_M [(S - 4n)f_4 + 3S^2 + 3f_3^2 + \frac{3}{4}|\nabla S|^2] \geq 0.\]

**Lemma 3.** Let \(M\) be a closed minimal hypersurface in \(S^{n+1}\). Then

\[(17) \quad \int_M |\nabla S|^2 \leq \frac{4n}{3n + 2} \int_M S^2(S - n).\]

**Proof.** By using the Schwartz inequality, the minimality condition, and the Schwarz inequality again, we obtain (cf. Schoen, Simon, and Yau [4])

\[|\nabla S|^2 = 4 \sum_k \left( \sum_{i,j} h_{ij} h_{ijk} \right)^2\]

\[= 4 \sum_k \left( \sum_i \lambda_i h_{iik} \right)^2 \leq 4S \sum_{i,k} h_{iik}^2\]

\[= 4S \sum_{i \neq k} h_{iik}^2 + 4S \sum_i h_{iii}^2\]

\[(18)\]

\[= 4S \sum_{i \neq k} h_{iik}^2 + 4S \sum_i \left( \sum_{j \neq i} h_{jji} \right)^2\]

\[\leq 4S \sum_{i \neq k} h_{iik}^2 + 4S(n - 1) \sum_{j \neq i} h_{jji}^2\]

\[= 4nS \sum_{i \neq j} h_{jji}^2.\]

On the other hand, by (5)

\[|\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2\]

\[= 3 \sum_{i \neq k} h_{iik}^2 + \sum_i h_{iii}^2 + \sum_{i,j,k \neq} h_{ijk}^2\]

\[(19)\]

\[\geq 3 \sum_{i \neq k} h_{iik}^2 + \sum_i h_{iii}^2\]

\[= 2 \sum_{i \neq k} h_{iik}^2 + \sum_i h_{iii}^2 + \sum_{i \neq k} h_{iik}^2.\]

From (18) and (19) we get

\[(20) \quad |\nabla S|^2 \leq \frac{4n}{n + 2} S|\nabla h|^2.\]
By the second formula of (8),

\[(21) \quad S|\nabla h|^2 = \frac{1}{4} \Delta S^2 - \frac{1}{2} |\nabla S|^2 - (n - S)S^2.\]

Thus (17) comes from (20) and (21).

Combining Lemma 2 with Lemma 3, we have

**Proposition 1.** Let \( M \) be a closed minimal hypersurface in \( S^{n+1} \). Then we have

\[(22) \quad \int_M \left( (S - 4n)f_4 + 3S^2 + 3f_3^2 + \frac{3n}{3n + 2} (S - n)S^2 \right) \geq 0.\]

In the case \( n = 3 \), \( f_4 = S^2/2 \), \( f_3 = 3 \det(h) \). We have

**Corollary 1.** Let \( M \) be a closed minimal hypersurface in \( S^4 \). Then we have

\[(23) \quad \int_M \left[ \frac{1}{2}S^2(S - 6) + 27(\det(h))^2 + \frac{9}{11} (S - 3)S^2 \right] \geq 0.\]

3. PROOF OF THE THEOREM

Let \( M \) be a closed minimal hypersurface in \( S^4 \). Near any given point \( p \), we can choose a local frame field \( e_1, \ldots, e_3 \) so that at \( p \), \( h_{ij} = \lambda_i \delta_{ij} \). By Gauss equations

\[(24) \quad R_{ii} = 2 - \lambda_i^2, \quad 1 \leq i \leq 3.\]

From assumption condition (2), we have

\[\lambda_i^2 \leq 2, \quad i = 1, 2, 3.\]

Therefore, we obtain

\[(25) \quad \prod_{i=1}^{3} (\lambda_i^2 - 2) = (\det(h))^2 + f_4 - S^2 + 4S - 8 = (\det(h))^2 - \frac{1}{2}S^2 + 4S - 8.\]

Combining (25) with (23), we obtain

\[(26) \quad \int_M \left[ \frac{1}{2}(S - 3)[\frac{29}{11}S^2 + 24S - 144] \right] \geq 0,\]

that is

\[(27) \quad \int_M \frac{29}{11}(S - 3)[(S + \frac{12}{19}(11 + \sqrt{440}))(S - \frac{12}{19}(-11 + \sqrt{440}))] \geq 0.\]

Since we assume \( \text{Ric}(M) \geq 0 \), i.e.,

\[(28) \quad -\sqrt{2} \leq \lambda_i \leq \sqrt{2}, \quad i = 1, 2, 3,\]

the minimality condition of \( M \) is

\[(29) \quad \lambda_1 + \lambda_2 + \lambda_3 = 0.\]
It is easily seen that the convex function \( S = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \) of three variables \( \lambda_1, \lambda_2, \lambda_3 \) subject to linear constraint conditions (28) and (29) attains its maximum when (after renumbering \( e_1, e_2, e_3 \) if necessary)

\[
\lambda_1 = -\lambda_2 = \sqrt{2}, \quad \lambda_3 = 0.
\]

Therefore, we have

\[
S \leq 4.
\]

On the other hand, since we also assume \( \text{Ric}(M) \leq 3/2 \), i.e., \( \lambda_i^2 \geq 1/2 \), \( i = 1, 2, 3 \), we have \( S \geq 3 \) from (29). In view of \( 3 \leq S \leq 4 \), we obtain from (27), \( S \equiv 3 \), i.e., \( M = C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3}) \) (see Lawson [2] or Chern, do Carmo, and Kobayashi [1]). We complete the proof of the Theorem.

**References**


**Institute of Mathematics, Academia Sinica, Beijing, 100080 People's Republic of China**

**Current address**: Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, People's Republic of China