ON THE DETERMINANT OF ELLIPTIC BOUNDARY VALUE PROBLEMS ON A LINE SEGMENT

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ABSTRACT. In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment \([0, T]\) with boundary conditions.

1. Introduction and summary of the results

In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment \([0, T]\) with boundary conditions. In order to state our results we introduce the following notation:

1. Denote by \(\mathcal{A} = \sum_{k=0}^{2n} a_k(x)D^k\) a differential operator, \(D = D_x = -i \frac{d}{dx}\), where the coefficients are complex-valued \(r \times r\) matrices depending smoothly on \(x\), \(0 < x < T\). The leading coefficient \(a_{2n}(x)\) is assumed to be nonsingular and to have \(\theta\) as a principal angle, i.e. \(R_\theta \cap \text{Spec} a_{2n}(x) = \emptyset\) for \(0 \leq x \leq T\), where 

\[
R_\theta := \{pe^{i\theta} \in \mathbb{C} \mid 0 < p < \infty\}.
\]

2. We impose boundary conditions of the form

\[
\ell_j u(T) = 0d, \quad m_j u(0) = 0 \quad (1 \leq j \leq n)
\]
where \(u \in C^\infty([0, T] ; \mathbb{C}^r)\) and \(\ell_j, m_j\) are differential operators of the form

\[
\ell_j := \sum_{k=0}^{\alpha_j} b_{jk} d_x^k, \quad m_j := \sum_{k=0}^{\beta_j} c_{jk} d_x^k \quad (d_x = \frac{d}{dx})
\]

such that \(b_{jk}, c_{jk}\) are constant \(r \times r\) matrices with \(b_{j\alpha_j} = c_{j\beta_j} = \text{Id}\) and such that the integers \(\alpha_j, \beta_j\) satisfy

\[
0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 2n - 1, \quad 0 \leq \beta_1 < \beta_2 < \cdots < \beta_n \leq 2n - 1.
\]

Example 1. Dirichlet boundary conditions: \(\alpha_D = \beta_D = (0, 1, \ldots, n - 1)\)

\[
b_{D,jk} = c_{D,jk} := \begin{cases} 
\text{Id} & \text{if } 1 \leq j \leq n, k = j - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

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Example 2. Neumann boundary conditions: \( \alpha_N = \beta_N = (n, n+1, \ldots, 2n-1) \)

\[
b_{N,j} = c_{N,j} := \begin{cases} 1 \text{d} & \text{if } 1 \leq j \leq n, k = n + j - 1, \\ 0 & \text{otherwise}. \end{cases}
\]

For convenience we write \( \alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = \sum_{j=1}^{n} \alpha_j \) and similarly \( \beta \) and \( |\beta| \). Boundary conditions of the above form are usually called separated. Let \( B = (B_{jk}) \) and \( C = (C_{jk}) \), \( 1 \leq j \leq 2n, \quad 0 \leq k \leq 2n - 1 \), be \( 2n \times 2n \) matrices whose entries are the following \( r \times r \) matrices

\[
B_{jk} = \begin{cases} b_{jk} & \text{if } 1 \leq j \leq n \text{ and } 0 \leq k \leq \alpha_j, \\ 0 & \text{otherwise}; \end{cases}
\]

\[
C_{jk} = \begin{cases} c_{j-n,k} & \text{if } n + 1 \leq j \leq 2n \text{ and } 0 \leq k \leq \beta_{j-n}, \\ 0 & \text{otherwise}. \end{cases}
\]

We denote by \( A = A_{B,C} \) the operator \( \mathcal{A} \) restricted to the space of smooth functions \( u : [0, T] \to \mathbb{C}' \) satisfying the boundary conditions (1.1).

(3) \( \zeta \)-regularized determinant \( \text{Det}_{\zeta} A \). In the case where \( A \) is not 1-1, define \( \text{Det}_{\zeta} A = 0 \). In the case \( A \) is 1-1, one proceeds as follows. As the coefficient \( a_{2n}(x) \) has \( \theta \) as a principal angle, there exists \( \varepsilon > 0 \) so that \( L(\theta - \varepsilon, \theta + \varepsilon) \cap \text{Spec} a_{2n}(x) = \emptyset, \quad 0 \leq x \leq T \), where \( L(\alpha, \beta) := \{ z \in \mathbb{C} \mid \alpha \leq \arg z \leq \beta \} \). Then the spectrum of \( A \), \( \text{Spec} A \), is discrete, \( \text{Spec} A = \{ \lambda_j, j \in \mathbb{N} \} \), \( |\lambda_j| \to \infty \), and \( \text{Spec} A \cap L(\theta - \varepsilon', \theta + \varepsilon') \) for any \( 0 < \varepsilon < \varepsilon' \) is finite.

If \( R_\theta \cap \text{Spec} A = \emptyset \), we define \( \zeta_{A, \theta}(s) = \sum_{j \geq 1} \lambda_j^{-s} = 
\]

\[
\text{Tr} A^{-s} \quad \text{where } s \in \mathbb{C}, \quad \text{Re} s > 1/2n \quad \text{and where the complex powers are defined with respect to the angle } \theta. \]

It is a well-known fact that \( \zeta_{A, \theta}(s) \) admits a meromorphic extension to \( \mathbb{C} \) with \( s = 0 \) being a regular point. According to Ray and Singer [RS] one defines \( \log \text{Det}_{\zeta} A := \lim_{\varepsilon \to 0} \zeta_{A, \theta}(s) \). If \( R_\theta \cap \text{Spec} A = \emptyset \), then choose \( \theta' \in (\theta - \varepsilon, \theta + \varepsilon) \) so that \( R_{\theta'} \cap \text{Spec} A = \emptyset \), and define \( \text{Det}_{\zeta} A := \text{Det}_{\zeta}(A) \).

It can be easily checked (cf. [BFK1]) that the definition is independent of the choise of \( \theta' \) in \( \theta' = (\theta - \varepsilon, \theta + \varepsilon) \).

(4) The fundamental matrix \( Y(x) = Y(x, \mathcal{A}) \). Denote by \( Y(x) = (y_{k\ell}(x)) \quad (x \in \mathbb{R}) \) the fundamental matrix for \( \mathcal{A} \). Note that \( Y(x) \) is a \( 2n \times 2n \) matrix whose entries \( y_{k\ell}(x) \) \( (0 \leq k, \ell \leq 2n - 1) \) are \( r \times r \) matrices defined by

\[
y_{k\ell}(x) := a_{k\ell}^x(x),
\]

where \( y_{k\ell}(x) \) denotes the solution of the Cauchy problem \( \mathcal{A} y_{k\ell}(x) = 0, \quad y_{k\ell}(0) = \delta_{k\ell} \text{Id} \). Of particular interest is the \( 2n \times 2n \) matrix \( Y(T) \), the evaluation of the fundamental matrix at \( x = T \).

(5) Introduce the quantities

\[
g_\alpha := \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right), \quad h_\alpha = \det \begin{pmatrix} w_1^{\alpha_1} & \cdots & w_n^{\alpha_1} \\ \vdots & \ddots & \vdots \\ w_1^{\alpha_n} & \cdots & w_n^{\alpha_n} \end{pmatrix}
\]

where \( w_1, \ldots, w_n \) denote the \( 2n \) th roots of \( (-1)^{n+1} \) with \( \text{Re} w > 0 \) given by \( w_k = \exp \left\{ \frac{2k-n-1}{2n} \pi i \right\} \). For a \( r \times r \) matrix \( a \) with principal angle \( \theta \) and eigenvalues \( \lambda_1, \ldots, \lambda_r \), denote \( (\det a)^{\theta}_{\alpha_\pi} = \prod_{j=1}^{r} |\lambda_j|^{\alpha_\pi} \exp \{ ig_\alpha \arg \lambda_j \} \) where \( \theta - 2\pi < \arg \lambda_j < \theta \).
Example 1. Dirichlet boundary conditions:

\[ g_{\alpha_D} = -n/4, \quad h_{\alpha_D} = h_n := \prod_{i \geq j} (w_i - w_j). \]

Example 2. Neumann boundary conditions:

\[ g_{\alpha_N} = n/4, \quad h_{\alpha_N} = (-1)^n h_n. \]

The main result of this paper is Theorem.

\[ \text{Det}_{\theta} A = K_{\theta} \exp \left\{ \frac{i}{2} \int_0^T \text{tr}(a_{2n-1}(x)a_{2n-1}(x)) \, dx \right\} \det(BY(T) - C) \]

where \( K_{\theta} \equiv K_{\theta}(\alpha, \beta) \) is given by

\[ K_{\theta} = ((-1)^{|\alpha_D|}(2n)^n h_{\alpha_D}^{-1} h_{\beta_D}^{-1})^\theta (\det a_{2n}(0))_{\theta}^{\frac{n}{2}} (\det a_{2n}(T))_{\theta}^{\frac{n}{2}}. \]

Example 1. Dirichlet boundary conditions:

\[ |\alpha_D| = \frac{n(n-1)}{2}, \quad K_{\theta} = ((-1)^{|\alpha_D|}(2n)^n h_{\alpha_D}^{-2})^\theta (\det a_{2n}(0))_{\theta}^{\frac{n}{4}} (\det a_{2n}(T))_{\theta}^{\frac{n}{4}}. \]

Example 2. Neumann boundary conditions:

\[ |\alpha_N| = \frac{n(n-1)}{2}, \quad K_{\theta} = ((-1)^{|\alpha_N|}(2n)^n h_{\alpha_N}^{-2})^\theta (\det a_{2n}(0))_{\theta}^{\frac{n}{4}} (\det a_{2n}(T))_{\theta}^{\frac{n}{4}}. \]

Corollary. \( \text{Det}_{\theta} A \) is a complex number independent of \( \theta \) up to multiplication with a \( 2n \)th root of unity.

Remark 1. In the formula above all terms except the matrix \( Y(T) \) are easily computable from the coefficients of \( \mathcal{A} \), \( \ell_i \) and \( m_j \). The matrix \( Y(T) \) requires the knowledge of the fundamental solutions. The matrix \( Y(T) \) and therefore \( \det(BY(T) - C) \) can be calculated numerically within arbitrary accuracy by solving a finite difference equation approximating \( \mathcal{A} \). So the determinant \( \text{Det}_{\theta} A \) can be calculated numerically within arbitrary accuracy.

Remark 2. Theorem is a companion of the corresponding result on the circle instead of the interval \([0, T]\) which was treated in an earlier paper [BFK1]. Again, the proof of Theorem relies on a deformation argument and explicit computations for certain special operators and special boundary conditions.

Remark 3. Introduce a spectral parameter \( \lambda \), and denote the fundamental matrix of \( \mathcal{A} + \lambda \) by \( Y(x, \lambda) = Y(x, \mathcal{A} + \lambda) \). One then verifies \( \det(BY(T; \lambda) - C) = 0 \) iff \( \text{Det}_{\theta}(A + \lambda) = 0 \), i.e. iff \( -\lambda \) is an eigenvalue of \( A = A_B, C \).

Remark 4. First results of the type described in Theorem are due to Dreyfus and Dym [DD] and to Forman [Fo1] (cf. also [Fo2]). Forman proved by different methods that the quotient \( \text{Det}_{\theta} A / \text{Det}(BY(T) - C) \) only depends on the principal and subprincipal symbols of \( \mathcal{A} \), and the principal symbol of the boundary operators \( \ell_j, m_j \ (1 \leq j \leq n) \). Our Theorem provides a formula for this quotient.
Remark 5. Analogous to results obtained in [BFK2], Theorem can be extended to the case where $\mathcal{A}$ is a pseudodifferential operator. The determinant $\text{Det}_\theta A$ can be written as a product of local invariants with a Fredholm determinant of a pseudodifferential operator of determinant class, canonically associated to $A$. The Fredholm determinant corresponds to $\det(BY(T) - C)$ in the case when $\mathcal{A}$ is a differential operator.

2. Auxiliary results

In this section we collect some auxiliary results needed for the proof of Theorem. First we introduce some additional notation. Denote by $EDO_{2n} \equiv EDO_{2n, r}$ the set of all elliptic differential operators $\mathcal{A}$ of order $2n$ on $[0, T]$ as introduced in Section 1. We identify $EDO_{2n}$ with the open set $\{(a_{2n}, \ldots, a_0) \in C^\infty([0, T], \text{End} C')^{2n+1} : \det(a_{2n}(x)) \neq 0, 0 \leq x \leq T\}$ of the Frechet space $C^\infty([0, T], \text{End} C')^{2n+1}$. Further define $EDO_{2n, \theta} := \{\mathcal{A} \in EDO_{2n} : \theta$ is principal angle for $a_{2n}\}$. Clearly $EDO_{2n, \theta}$ is an open connected subset in $EDO_{2n}$.

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 2n - 1$, we introduce the space $BDO_\alpha$ of operators used to define the boundary conditions:

$$BDO_\alpha := \{B = (b_{jk})_{0 \leq j, k \leq 2n-1} : b_{jk} \in \text{End} C', b_{ja_j} = \text{Id}, b_{jk} = 0 \text{ if } k \geq \alpha_j + 1\}.$$ 

Given $\alpha, \beta$, we introduce the space

$$EDO_{2n; \alpha; \beta} := \{A_B, C : \mathcal{A} \in EDO_{2n}, B \in BDO_\alpha, C \in BDO_\beta\}$$

where $A_B, C$ is the restriction of $\mathcal{A}$ to the subspace of functions $u \in C^\infty([0, T]; C')$ satisfying the boundary conditions defined by $B$ and $C$. Similarly introduce $EDO_{2n, \theta; \alpha; \beta} := \{A_B, C \in EDO_{2n; \alpha; \beta} : \mathcal{A} \in EDO_{2n, \theta}\}$. Observe that $\{A_B, C \in EDO_{2n; \alpha; \beta} : A_B, C \text{ is 1-1}\}$ is open.

Further, denote by $EDO_{2n; \alpha; \beta}$ the open subset of $EDO_{2n; \alpha; \beta} \times S^1$ consisting of pairs $(A_B, C, \theta)$ with $A_B, C \in EDO_{2n; \alpha; \beta}$. As in [BFK1] we have the following

Proposition 2.1. (1) $\text{Det}_\theta(A_B, C)$ is a smooth function on $EDO_{2n; \alpha; \beta}$ and is locally constant in $\theta$.

(2) $\text{Det}_\theta(A_B, C)$ is holomorphic when considered as a function on the open subset of injective operators in $EDO_{2n; \alpha; \beta}$.

(3) $\det(BY(T, \mathcal{A}) - C)$ is holomorphic on $EDO_{2n} \times BDO_\alpha \times BDO_\beta$.

Observe that a necessary and sufficient condition for $A_B, C$ to have zero as an eigenvalue is that $\det(BY(T) - C) = 0$, which in view of Proposition 2.1 (3) implies that the subsets of $EDO_{2n; \alpha; \beta}$ and $EDO_{2n; \alpha; \beta}$ consisting of injective operators are open (as we already noticed) and connected, and therefore, $EDO_{2n; \alpha; \beta}$ is open and connected as well.

Let $s : [0, T] \to GL(C')$ be a smooth map. Given $\mathcal{A} \in EDO_{2n}$ and boundary operators $\ell_j, m_j$ ($1 \leq j \leq n$) introduce $\mathcal{A}_1 := s(x)^{-1}\mathcal{A}s(x), \ell_{ij} := s(T)^{-1}\ell_j s(x) |_{x=T}$, and $m_{ij} := s(0)^{-1}m_j s(x) |_{x=0}$. Denote by $(B_{ijk})$ and $(C_{ijk})$ the matrices introduced in Section 1 corresponding to the boundary operators $(\ell_{ij}, m_{ij})_{1 \leq i \leq n}$ and write $Y_1(x) = Y(x, \mathcal{A}_1)$ for short.

Proposition 2.2. $\det(B_1Y_1(T) - C_1) = (\det s(0)s(T)^{-1})^n \det(BY(T) - C)$. 

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Proof. Let \( L = L(x) \) be a \( 2n \times 2n \) matrix with entries \( L_{k\ell} \) which are the following \( r \times r \) matrices (\( 0 \leq k, \ell \leq 2n - 1 \))

\[
L_{k\ell} := \binom{k}{\ell} d_{s-\ell} s(x) \quad \text{if } k \geq \ell; \quad L_{k\ell} = 0 \quad \text{if } k < \ell.
\]

Thus we obtain

\[
B_1 = \text{diag} (s(T)^{-1}, \ldots, s(T)^{-1})BL(T)
\]

where \( \text{diag} (s(T)^{-1}, \ldots, s(T)^{-1}) \) is a \( 2n \times 2n \) diagonal matrix whose entries on the diagonal are all equal to the \( r \times r \) matrix \( s(T)^{-1} \). Similarly, one obtains

\[
C_1 = \text{diag} (s(0)^{-1}, \ldots, s(0)^{-1})CL(0).
\]

Further, by a straightforward computation, \( Y_1 \) is given by

\[
Y_1(x) = L(x)^{-1} Y(x)L(0).
\]

Thus

\[
B_1 Y_1(T) - C_1 = \text{diag} (s(T)^{-1}, \ldots, s(T)^{-1}, s(0)^{-1}, \ldots, s(0)^{-1}) \cdot [BY(T) - C]L(0).
\]

Now observe that \( \det L(0) = (\det s(0))^{2n} \) as \( L(0) \) is lower triangular with diagonal entries all equal to the \( r \times r \) matrix \( s(0) \). This implies that

\[
\det (B_1 Y_1(T) - C_1) = (\det s(0)s(T)^{-1})^n \det (BY(T) - C).
\]

Next consider for \( A = A_B, c \in EDO_{2n; \theta; \alpha; \beta} \) and \( \Phi \in C^\infty([0, T], GL_r(\mathbb{C})) \) the generalized \( \zeta \)-function \( \zeta_{\Phi, A; \theta}(s) := \text{tr} \Phi A_{\theta}^{-s} \). Again this is a function which is holomorphic in \( \text{Res } s > \frac{1}{2n} \) and has a meromorphic extension to the whole complex plane. Moreover \( s = 0 \) is a regular point. Recall that we have introduced \( g_\alpha := \frac{1}{2} (\frac{\lvert \alpha \rvert}{n} - n + \frac{1}{2}) \), and similarly \( g_\beta \).

Proposition 2.3.

\[
(2.1) \quad \zeta_{\Phi, A; \theta}(0) = g_\beta \text{tr} \Phi(0) + g_\alpha \text{tr} \Phi(T)
\]

As an immediate consequence we obtain

Corollary 2.4. \( \zeta_{A; \theta}(0) = r(g_\alpha + g_\beta) = r(\frac{\lvert \alpha \rvert + \lvert \beta \rvert}{2n} - n + 1) \).

Proof (Proposition 2.3). We first prove that there are numbers \( \tilde{g}_\alpha, \tilde{g}_\beta \in \mathbb{C} \) which only depend on \( \alpha \) and \( \beta \) respectively such that (2.1) holds. The actual values of \( \tilde{g}_\alpha, \tilde{g}_\beta \) are computed at the end of section 3 by considering the case \( \Phi(x) \equiv K \) with \( K > 1 \), \( \mathcal{A} = D^n + \lambda, \theta = \pi \). In the course of the proof we use a number of results due to Seeley [Se1,2]. For the convenience of the reader we partly keep Seeley’s notation. For simplicity, we write \( \zeta(s) = \zeta_{\Phi, A; \theta}(s) \). According to [Se2], the value \( \zeta(0) \) consists of a sum of two terms, \( \zeta(0) = I + II \) where \( I \) represents the contribution to \( \zeta(0) \) of the resolvent of \( \mathcal{A} - \lambda \) and \( II \) represents a correction term due to the boundary conditions. According to [BFK1, p. 8],

\[
I = -\frac{e^{i\theta}}{4\pi n} \sum_{\tau = \pm 1} \int_0^T dx \int_0^\infty dr \text{tr} \{\Phi(x)c_{-2n-1}(x, \tau, re^{i\theta})\}
\]

where \( c_{-2n-1}(x, \tau, \lambda) \) comes from the expansion of the symbol

\[
r(x, \tau, \lambda) = c_{-2n}(x, \tau, \lambda) + c_{-2n-1}(x, \tau, \lambda) + \cdots
\]
of the parametrix for $\mathcal{A} - \lambda = (a_{2n}(x)D^{2n} - \lambda) + \sum_{j=0}^{2n-1} a_j(x)D^j$ and is given by
\[
c_{-2n-1}(x, \tau, \lambda) = -\tau^{2n-1}c_{-2n}a_{2n-1}c_{-2n} - i2n\tau^{4n-1}c_{-2n}a_{2n}c_{-2n} \left( \frac{d}{dx} a_{2n} \right) c_{-2n},
\]
where $c_{-2n} \equiv c_{-2n}(x, \tau, \lambda) = (a_{2n}(x)x^{2n} - \lambda)^{-1}$.

As in [BFK1], Proposition 2.8, in view of the fact that $c_{-2n-1}$ is odd in $\tau$, we conclude $I = 0$. From [Se2], p. 968, it follows that $II$ is of the form
\[
II = \text{tr} \left\{ \Delta_0(0)\Phi(0) + \Delta_T(0)\Phi(T) \right\}
\]
where $\Delta_0(s)$ and $\Delta_T(s)$ are smooth functions described below. Let us first consider the scalar case, $r = 1$. In first approximation the kernel $r(x, y, \lambda)$ of $(A_B, c - \lambda)^{-1}$ is given by
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (a_{2n}(x)\tau^{2n} - \lambda)^{-1}e^{i(x-y)\tau} d\tau + r_0(x, y, \lambda) + r_T(x, y, \lambda)
\]
where $r_0(x, y, \lambda)$ and $r_T(x, y, \lambda)$ are correction terms so that in first approximation $r(x, y, \lambda)$ satisfies the boundary conditions at $x = 0$ and $x = T$. Let us explain how to obtain $r_0(x, y, \lambda)$; for $r_T(x, y, \lambda)$ one proceeds in a similar fashion. Consider the boundary value problem
\[
(aD^{2n} - \lambda)u = 0
\]
with the boundary condition
\[
\lim_{x \to \infty} u(x) = 0; \quad D^\beta_i u(0) = -(a\tau^{2n} - \lambda)^{-1}\tau^\beta_i e^{-iy\tau}
\]
where $a = a_{2n}(0)$ and $D = \frac{1}{i} \frac{d}{dx}$. The solution $u(x) = u(x, \tau, y, \lambda)$ of the boundary value problem (2.2)-(2.3) is given by $u(x) = \sum_{\nu=1}^{n} w_{\nu}e^{i(x-\lambda/a)^{1/2n}w_{\nu}}$ where $w_{\nu}$ ($1 \leq \nu \leq n$) are the 2n th roots of $-1$ with strictly positive imaginary part and where $(-\lambda/a)^{1/2n} = (-\lambda/a)^{1/2n} = (|\lambda/a|^{2n} e^{i(\theta - \pi - \arg a)/2n}$ with $\lambda = |\lambda|e^{i\theta}$ and $\theta - 2\pi < \arg a < \theta$. The coefficients $u_{\nu} = u_{\nu}(\tau, y, \lambda)$ are then determined by (2.3)
\[
\sum_{\nu=1}^{n} u_{\nu} \left( -\lambda \over a \right)^{\beta_i/2n} w_{\nu}^{\beta_i} = -\tau^\beta_i (a\tau^{2n} - \lambda)^{-1}e^{-iy\tau}.
\]
Thus
\[
u_{\nu} = -\sum_{j=1}^{n} \mathcal{H}_{\nu j}(-\lambda/a)^{-\beta_j/2n} \tau^\beta_j (a\tau^{2n} - \lambda)^{-1}e^{-iy\tau}
\]
with $\mathcal{H}_{\nu j}$ defined by
\[
\sum_{j=1}^{n} \mathcal{H}_{\nu j} w_{k}^{\beta_j} = \delta_{\nu k}.
\]
The term $r_0(x, y, \lambda)$ is then given by
\[
r_0(x, y, \lambda) = \sum_{\nu=1}^{n} e^{i(x-\lambda/a)^{1/2n}w_{\nu}} \sum_{j=1}^{n} \mathcal{H}_{\nu j} \frac{1}{i} (-\lambda/a)^{-\beta_j/2n} f
\]
where $\mathcal{J}$ is the sum of residues

$$
\mathcal{J} = \sum_{k=1}^{n} \text{Res}_{x_k = (-\lambda/a)^{1/2n}} \{ \tau^\beta_j (a \tau^{2n} - \lambda)^{-1} e^{-iy\tau} \}
$$

of $\tau^\beta_j (a \tau^{2n} - \lambda)^{-1} e^{-iy\tau}$ in the lower half plane. One obtains

$$
\mathcal{J} = \sum_{k=1}^{n} ((-\lambda/a)^{1/2n} \overline{w_k})^{\beta_j - (2n-1)} \frac{1}{2na} \exp\{ -y(-\lambda/a)^{1/2n} \overline{w_k} \}.
$$

Summarizing one obtains

$$
 r_0(x, y, \lambda) = \frac{i}{2na} (\lambda/a)^{-(2n-1)/2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} \overline{w_k}^{\beta_j+1} \exp\{ i(-\lambda/a)^{1/2n}(x \nu - y \overline{w_k}) \}.
$$

Following Seeley, we now define for $\Re s > 0$

$$
(2.5) \quad \Delta_0'(s) := \int_0^{T/2} dx \frac{1}{2\pi i} \int_{\Gamma_0} d\lambda \lambda^{-s} r_0(x, x, \lambda)
$$

where $\Gamma_0$ is the contour that goes from $\infty$ to 0 along the lower side of ray $\{ re^{i\theta} : r > 0 \}$, goes around the origin and then returns to $\infty$ along the upper side of the ray $\{ re^{i\theta} : r > 0 \}$. By a standard computation,

$$
\frac{1}{2\pi i} \int_{\Gamma_0} d\lambda \lambda^{-s} r_0(x, x, \lambda) = a^{-s} e^{-i\pi s} \frac{\sin \pi s}{\pi} \Gamma(1 - 2ns) \sum_{\nu, j, k} \mathcal{H}_{\nu j} \overline{w_k}^{\beta_j+1} ((\nu \nu - \overline{w_k})x)^{-1+2ns}
$$

and therefore

$$
\Delta_0'(0) = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} \frac{\overline{w_k}^{\beta_j+1}}{\nu \nu - \overline{w_k}}.
$$

In the case $r > 2$, we first treat the case where all eigenvalues of $a_{2n}(0)$ are different which can be easily reduced to scalar case $r = 1$. By a continuity argument we then conclude that

$$
(2.6) \quad \tilde{g}_\beta = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j}(\beta) \overline{w_k}^{\beta_j+1} (\nu \nu - \overline{w_k})^{-1}
$$

where $\mathcal{H}_{\nu j} = \mathcal{H}_{\nu j}(\beta)$ are determined by (2.4). Similarly one obtains

$$
\tilde{g}_\alpha = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j}(\alpha) \overline{w_k}^{\alpha_j+1} (\nu \nu - \overline{w_k})^{-1}.
$$

### 3. Proof of Theorem 1

For the proof of Theorem we need two deformation results. The first one is the analogue of Proposition 3.1 in [BFK1] and proved in a similar way (cf. also [DD] and [Fo1]).
Proposition 3.1. Suppose \( \mathcal{A} = \sum_{k=0}^{2n} a_k(x)D^k \) and \( \mathcal{A}' = \sum_{k=0}^{2n} a'_k(x)D^k \) are in \( EDO_{2n; \theta} \) with \( a_{2n} = a'_{2n} \) and \( a_{2n-1} = a'_{2n-1} \). Then, for \( B \in BDO_\alpha \) and \( C \in BDO_\beta \)

\[
\text{Det}_\theta(AB, C) \det(BY(T; \mathcal{A}') - C) = \text{Det}_\theta(A'B', C') \det(B'Y(T; \mathcal{A}') - C').
\]

The second result concerns a deformation of the boundary conditions. Consider boundary operators \( 1 \leq j \leq n, dx = \frac{d}{dx} \)

\[
\ell_j = \sum_{k=0}^{\alpha_j} b_{jk}d_x^k, \quad m_j = \sum_{k=0}^{\beta_j} c_{jk}d_x^k; \quad b_{j\alpha_j} = c_{j\beta_j} = \text{Id}
\]

and

\[
\ell'_j = d_x^{\alpha_j}, \quad m'_j = d_x^{\beta_j}.
\]

Form the matrices \( B, C \) and \( B', C' \) as in Section 1.

Proposition 3.2. Fix \( \mathcal{A} \in EDO_{2n; \theta} \). Then

\[
\text{Det}_\theta(A'B', C') \det(B'Y(T) - C') = \text{Det}_\theta(A'B, C) \det(BY(T; \mathcal{A}) - C).
\]

Proof. Without loss of generality we may assume that both \( AB, C \) and \( AB', C' \) are injective. Note that \( \{ A\tilde{B}, \tilde{C} : A\tilde{B}, \tilde{C} \text{ is 1-1}, \tilde{B} \in BDO_\alpha, \tilde{C} \in BDO_\beta \} \) is arcwise connected in \( BDO_\alpha \times BDO_\beta \). Define, for \( 0 < t < 1 \),

\[
\ell_{ij} = d_x^{\alpha_{ij}} + t \sum_{k=0}^{\alpha_{ij}-1} b_{jk}d_x^k, \quad m_{ij} = d_x^{\beta_{ij}} + t \sum_{k=0}^{\beta_{ij}-1} c_{jk}d_x^k
\]

such that, with \( B_t \) and \( C_t \) the corresponding matrices in \( BDO_\alpha \) and \( BDO_\beta \),

\[
\text{(3.1)} \quad A_{B_t}, C_t \text{ is 1-1 for } 0 \leq t \leq 1; \quad \text{(3.2)} \quad (B_0, C_0) = (B', C'), \quad (B_1, C_1) = (B, C).
\]

Introduce

\[
w(t) := \frac{d}{dt} \frac{\text{Det}_\theta(A_{B_t}, C_t)}{\text{Det}_\theta(A_B, C)}, \quad \delta(t) := \frac{d}{dt} \frac{\text{det}(B_tY(T) - C_t)}{\text{det}(B_tY(T) - C)}.
\]

The claimed result follows once we show that \( w(t) = \delta(t) \) \( (0 \leq t \leq 1) \). Let us first consider \( \delta(t) \). Denote by \( P_t \) the Poisson operator corresponding to the boundary value problem defined by \( (B_t, C_t) \). Then \( P_t \) is given by \( P_t = Y(x)(B_tY(T) - C_t)^{-1} \) and

\[
\delta(t) = \text{tr} \{(B_tY(T) - C_t)(B_tY(T) - C_t)^{-1}\}
\]

\[
= \text{tr} \{(\ell_{ij}, m_{ij})_{1 \leq j \leq n}P_t\}
\]

when \( \cdot = \frac{d}{dt} \) and \((\ell_{ij}, m_{ij})_{1 \leq j \leq n}\) is the operator associating to a section \( u \) the boundary values \((\ell_{ij}u(T), m_{ij}u(0))_{1 \leq j \leq n}\).

Next we consider \( w(t) \); with the notation \( A_t = A_{B_t}, C_t \),

\[
w(t) = F.p.s=0 \text{tr} (A_t^{-1}A_t^{-1-s})
\]

where \( F.p.s=0 \) denotes the finite part at \( s = 0 \). In order to evaluate \( A_t^{-1}A_t' = -(A_t^{-1})'A_t \), consider for a fixed section \( u : [0, T] \rightarrow C' \) the section \( v_t := A_t^{-1}u \), i.e. \( v_t \) satisfies

\[
\mathcal{A} v_t = u, \quad B_t v_t(T) = 0, \quad C_t v_t(0) = 0.
\]
Taking derivatives with respect to \( t \) we obtain
\[
\mathcal{A} v'_i = 0, \quad t_{ij} v'_i(T) = -t_{ij} v_i(T), \quad m_{ij} v'_i(0) = -m_{ij} v_i(0) \quad (1 \leq j \leq n).
\]
Thus \( v'_i = -P_i(t_{ij} v_i(T), m_{ij} v_i(0))_{1 \leq j \leq n} \) where \( P_i \) again denotes the Poisson operator. Thus we have proved that \( (A_i^{-1})' = -P_i(t_{ij}, m_{ij})_{1 \leq j \leq n} A_i^{-1} \). Note that \( (A_i^{-1})' A_i = -P_i(t_{ij}, m_{ij})_{1 \leq j \leq n} \) is a singular Green's operator of order \( \leq -2 \) and then of trace class. Thus
\[
w(\tau)' = \text{tr} P_i(t_{kj}, m_{ij})_{1 \leq j \leq n}. \quad \square
\]

**Proof of Theorem.** We have to prove that
\[
f_\theta(A_B, c) := \text{Det}_\theta(A_B, c) - K_\theta \exp \left\{ \frac{i}{2} \int_0^T \text{tr} (a_{2n}(x)^{-1} a_{2n-1}(x)) \, dx \right\}
\cdot \det (B Y(T) - C)
\]
vanishes identically on \{ \( A_B, c \in EDO_{2n}; \theta : A_B, c \) is 1-1 \}. First observe that it suffices to consider the case \( \theta = \pi \): For \( \mathcal{A} \in EDO_{2n}; \theta \), \( e^{i(\pi - \theta)} \mathcal{A} \in EDO_{2n}; \pi \) we have \( \log \text{Det}_\pi (e^{i(\pi - \theta)} A_B, c) = \log \text{Det}_\theta (A_B, c) + \zeta_{A, \theta}(0) \log e^{i(\pi - \theta)} \) and \( \log K_\theta (e^{i(\pi - \theta)} \mathcal{A}) = \log K_\pi (t) + r(g_\theta + g_\alpha) i(\pi - \theta) \); thus Corollary 2.4 allows to conclude the result as soon as we check it for \( \theta = \pi \).

To make writing easier, let \( f = f_\pi, \quad K = K_\pi, \quad \theta = \pi \).

**Deformation 1.** Consider the factorization \( \mathcal{A} = a_{2n}(D^{2n} + \mathcal{H}) \) where \( \mathcal{H} \) is a differential operator with ord \( \mathcal{H} \leq 2n - 1 \). Consider the 1-parameter family (0 \( \leq t \leq 1 \))
\[
\mathcal{A}_t := \alpha_t(D^{2n} + \mathcal{H}), \quad A_t := A_t, B, c
\]
when \( \alpha_t(x) = ta_{2n}(x) + (1 - t) \).

Clearly \( \theta = \pi \) is a principal angle for \( \alpha_t \) and \( A_t \) is 1-1 for 0 \( \leq t \leq 1 \). Moreover \( A_t' = (a_{2n}(x) - 1)(ta_{2n}(x) + (1 - t))^{-1} A_t \). Thus, with \( w(t) = \log \text{Det}_\pi A_t \) and Proposition 2.3
\[
w(t)' = F_{-p_{-s}} \text{tr} \left( (a_{2n}(x) - 1)(ta_{2n}(x) + (1 - t))^{-1} A(t)^{-1} \right)
= g_\beta \text{tr} [(a_{2n}(0) - 1)(ta_{2n}(0) + (1 - t))^{-1}]
+ g_\alpha \text{tr} [(a_{2n}(T) - 1)(ta_{2n}(T) + (1 - t))^{-1}]
= \frac{d}{dt} \left\{ g_\beta \log \det (ta_{2n}(0) + (1 - t)) \right\}
+ g_\alpha \log \det (ta_{2n}(T) + (1 - t)) \}.
\]
Thus
\[
\log \text{Det}_\pi A_1 - \log \text{Det}_\pi A_0 = \int_0^1 w(t)' \, dt = g_\beta \log \det (a_{2n}(0)) + g_\alpha \log \det (a_{2n}(T)).
\]
Hence we may and will assume that \( a_{2n}(x) \equiv \text{Id} \).

**Deformation 2.** Define \( s \in C^\infty([0, T]; \text{End} \, \mathbb{C}) \) by
\[
\frac{d}{dx} s(x) = i \frac{1}{2n} a_{2n-1}(x) s(x) \quad (0 \leq x \leq T); \quad s(0) = \text{Id}.
\]
Observe that \( \det(s(x)) = \exp\{\frac{1}{2n} \int_0^x \text{tr}(a_{2n-1}(y))dy\} \neq 0 \) for \( 0 \leq x \leq T \) and therefore \( s(x) \in \text{GL}_r(\mathbb{C}) \). Now consider \( \mathcal{A} := s(x)^{-1} s(x) \) and boundary conditions defined by \( B_1, C_1 \) (cf. Proposition 2.2). Then \( \text{Det}_x(\mathcal{A}) = \text{Det}_x(A) \) as the spectrum of \( A \) and the operator \( A_1 \), defined by \( \mathcal{A} \) and boundary conditions \( (B_1, C_1) \) do coincide. By Proposition 2.2,

\[
\det(B_1 Y(T) - C_1) = (\det s(T))^{-n} \det(B Y(T) - C).
\]

As we have noted above, \( \det s(T) = \exp\{\frac{1}{2n} \int_0^T \text{tr}(a_{2n-1}(y))dy\} \). Finally note that \( \mathcal{A} \) is of the form

\[
\mathcal{A} = D^{2n} + \sum_{k=0}^{2n-2} a_{1k}(x) D^k
\]

and then we may and will assume that for \( \mathcal{A}, a_{2n}(x) \equiv 1 \) and \( a_{2n-1}(x) \equiv 0 \).

**Deformation 3.** Applying Proposition 3.1 and Proposition 3.2 we conclude that it remains to prove that \( f(AB_1C) = 0 \) for \( \mathcal{A} = D^{2n} + \lambda \) and \( B, C \) given by

\[
\ell_j = d_{2j}^\alpha, \quad m_j = d_{2j}^\beta \quad (1 \leq j \leq n)
\]

where \( \lambda \) is chosen positive and sufficiently large so that \( A_{B,C} \) is \( 1-1 \). This is verified by an explicit computation. To make writing easier we restrict ourselves to that case \( r = 1 \). However, to obtain the explicit formulas for \( g_\alpha \) and \( g_\beta \) we consider \( \mathcal{A} = \rho D^{2n} + \lambda \) with \( \rho > 1 \). Denote by \( Y(x, \lambda) \) the fundamental matrix for \( \rho D^{2n} + \lambda \). For \( \lambda > 0 \), let \( \mu = (\frac{\lambda}{\rho})^{1/2n} \). Then, with \( w_k := \exp\{i\frac{2k-n-1}{2n}\pi\} \), \( Y(x, \lambda) \) is equal to

\[
\begin{pmatrix}
\mu w_1 e^{\mu w_2} & \cdots & \mu w_1 e^{\mu w_{2n}} \\
\mu w_1 e^{\mu w_2} & \cdots & \mu w_2 e^{\mu w_{2n}} \\
\vdots & \ddots & \vdots \\
(\mu w_1)^{2n-1} e^{\mu w_{2n}} & \cdots & (\mu w_2)^{2n-1} e^{\mu w_{2n}}
\end{pmatrix}^{-1}
\]

Further define \( B = (B_{jk}), C = (C_{jk}) \) by

\[
B_{jk} = \begin{cases}
1 & \text{if } 1 \leq j \leq n \text{ and } k = \alpha_j, \\
0 & \text{otherwise};
\end{cases}
\]

\[
C_{jk} = \begin{cases}
1 & \text{if } n+1 \leq j \leq 2n \text{ and } k = \beta_{j-n}, \\
0 & \text{otherwise}.
\end{cases}
\]

We have to show that

\[
(3.4) \quad \text{Det}_x((\rho D^{2n} + \lambda)_{B,C}) = (-1)^{j\beta}(2^n)^n (h_\alpha h_\beta)^{-1} \rho^{g_\alpha + g_\beta} \det(B Y(T, \lambda) - C).
\]

For that purpose we introduce

\[
w(\lambda) := \log \text{Det}_x((\rho D^{2n} + \lambda)_{B,C}),
\]

\[
\delta(\lambda) := \log \det(B Y(T; \lambda) - C).
\]

As \( n \geq 1 \), we know from Proposition 3.1 that \( \frac{d}{d\lambda} w(\lambda) = \frac{d}{d\lambda} \delta(\lambda) \). Therefore it suffices to consider the asymptotics of \( w(\lambda) \) and \( \delta(\lambda) \) as \( \lambda \to +\infty \).

First recall from [Fr] (cf. also [Vo]) that \( w(\lambda) \) admits an asymptotic expansion of the form \( \sum_{k=-1}^\infty p_k \lambda^{-k/n} + \sum_{j=0}^\infty q_j \lambda^{-1} \log \lambda \) with the property that \( p_0 = 0 \). To find the asymptotics of \( \delta(\lambda) \) as \( \lambda \to +\infty \), write \( Y(T, \lambda) \) in the form

\[
Y(T; \lambda) = L W E (L W)^{-1}
\]
where \( L = \text{diag}(1, \mu, \mu^2, \ldots, \mu^{2n-1}) \), \( E := \text{diag}(e^{\mu w_1 T}, \ldots, e^{\mu w_{2n} T}) \) and

\[
W = \begin{pmatrix}
1 & \cdots & 1 \\
\omega & \cdots & \omega \\
\vdots & \ddots & \vdots \\
\omega & \cdots & \omega
\end{pmatrix}_{n \times n}.
\]

Thus \( \delta(\lambda) = \log(\det W^{-1} L^{-1}) + \log(\det(BLWE - CLW)) \). Observe that the \((j, k)\)th coefficient of the matrix \( BLWE - CLW \) is of the form \( e^{\omega T} f_{jk}(\mu) + g_{jk}(\mu) \) where \( f_{jk}(\mu) \) and \( g_{jk}(\mu) \) are rational functions of \( \mu \). We conclude that, with \( \Omega = \sum_{j=1}^{n} w_j = \sum_{j=1}^{n} \Re w_j \),

\[
\log \det(BLWE - CLW) = \mu \Omega T + \log \det(BLW \begin{pmatrix}
\text{Id}_n & 0 \\
0 & 0
\end{pmatrix} - CLW \begin{pmatrix}
0 & 0 \\
0 & \text{Id}_n
\end{pmatrix}) + e(\lambda)
\]

where \( \lim e(\lambda) = 0 \). The matrix \( BLW \begin{pmatrix}
\text{Id}_n & 0 \\
0 & 0
\end{pmatrix} - CLW \begin{pmatrix}
0 & 0 \\
0 & \text{Id}_n
\end{pmatrix} \) is of the form

\[
\begin{pmatrix}
F^{(1)} & 0 \\
0 & F^{(2)}
\end{pmatrix}
\]

where \( F^{(i)} \) are \( n \times n \) matrices given by \((1 \leq j, k \leq n) \)

\[
F^{(1)}_{jk} := \mu^{\alpha_j} w_{n+k}^{\alpha_j}, \quad F^{(2)}_{jk} := -\mu^{\beta_j} w_{n+k}^{\beta_j} = (-1)^{\beta_j+1} \mu^{\beta_j} w_{n+k}^{\beta_j}
\]

where we used that \( w_{n+k} = -w_k \). Therefore, with \(|\alpha| = \sum_{j=1}^{n} \alpha_j, \ |\beta| = \sum_{j=1}^{n} \beta_j \)

\[
\det(BLW \begin{pmatrix}
\text{Id}_n & 0 \\
0 & 0
\end{pmatrix} - CLW \begin{pmatrix}
0 & 0 \\
0 & \text{Id}_n
\end{pmatrix}) = \mu^{|\alpha|} \det(w_{n+k}^{\alpha_j}) \mu^{|\beta|} (-1)^{|\beta| + n} \det(w_{n+k}^{\beta_j}).
\]

In view of the fact that \( \det L^{-1} l_{\lambda} = \prod_{j=0}^{2n-1} (\frac{1}{\lambda} - j/2n) = \rho^{2n-1} \), this implies that the 0th order coefficient of the asymptotic expansion of \( \delta(\lambda) \) for \( \lambda \to \infty \) is of the form

\[
\delta_{+\infty} := \det L^{-1} l_{\lambda} = \log(\det(W^{-1}) \det(w_k^{\alpha_j})(-1)^{|\beta| + n} \det(w_k^{\beta_j}) \rho^{-(|\alpha| + |\beta|)/2n})
\]

\[
= \log \rho^{2n-1} - \log \rho^{(|\alpha| + |\beta|)/2n} + \log((-1)^{|\beta| + n} \det(W^{-1}) \det(w_k^{\alpha_j}) \det(w_k^{\beta_j})).
\]

where \( h_\alpha = \det(w_k^{\alpha_j}), \ h_\beta = \det(w_k^{\beta_j}) \).

By a straightforward computation we have \( \det W = (-1)^n (2n)^n \) and therefore

\[
(3.5) \quad w(\lambda) = \delta(\lambda) - \delta_{+\infty} = \delta(\lambda) + \log((-1)^{|\beta|} (2n)^n h_\alpha^{-1} h_\beta^{-1} \rho^{(\frac{|\alpha|}{2n} - \frac{n}{2} + \frac{1}{4} + \frac{|\alpha|}{2n} - \frac{n}{2} + \frac{1}{4})}.
\]

The claim (3.4) then follows from the following.

**Lemma 3.3.** \( \tilde{g}_\alpha = \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right) \).

**Proof.** In view of Proposition 2.3 we obtain from (3.5) in the case \( \alpha = \beta \)

\[
2 \tilde{g}_\alpha = 2 \left( \frac{|\alpha|}{2n} - \frac{n}{2} + \frac{1}{4} \right) \quad \text{or} \quad \tilde{g}_\alpha = \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right). \quad \Box
\]
REFERENCES


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