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AN INVERSE PROBLEM IN BIFURCATION THEORY, III

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ABSTRACT. This note is concerned with a nonlinear boundary value problem for a simple ordinary differential equation. A sufficient condition for the second bifurcating curve to determine a nonlinear term uniquely is obtained.

Consider the boundary value problem

$$(1)\qquad \begin{cases} \dfrac{d^2u}{dx^2} + \lambda u = g(u), & 0 < x < \pi, \\ u(0) = u(\pi) = 0, \end{cases}$$

where g is a real function of class $C^1(\boldsymbol{R})$ with $g(0) = g'(0) = 0$. By a solution of (1) we mean a pair $(\lambda, u) \in \boldsymbol{R} \times C^2[0, \pi]$. The assumption $g(0) = g'(0) = 0$ means that $g(u)$ is smaller than the term λu near $u = 0$, that is, $g(u) = o(u)$ as $u \to 0$. Therefore the linearlized problem of (1) near the trivial solution $u \equiv 0$ is:

$$\begin{cases} \dfrac{d^2u}{dx^2} + \lambda u = 0, & 0 < x < \pi, \\ u(0) = u(\pi) = 0. \end{cases}$$

It is clear that the solution set of this problem consists of the solutions $(\lambda, 0)$, $\lambda \in \boldsymbol{R}$, and the solutions $(n^2, h \sin nx)$ for each $n = 1, 2, \ldots$ and any $h \in \boldsymbol{R} \setminus \{0\}$. General theorems (see e.g. [1, 5]) in bifurcation theory assert that the solution branches of nonlinear problems bifurcate from eigenvalues of the linearlized problems. Therefore each solution branch of (1) bifurcates at the point $(n^2, 0)$. In view of this phenomenon, the solution branch of (1) emanating from $(n^2, 0)$ will henceforth be called the nth bifurcating curve and denoted by $\Gamma_n(g)$. More precisely, the sets $\Gamma_n(g)$, $n = 1, 2, \cdots$, are defined by

(2)

$$\Gamma_n(g) := \Big\{(\lambda, h) \in \boldsymbol{R}^2 \,|\, \text{there exists a solution } (\lambda, u) \in \boldsymbol{R} \times C^2[0, \pi] \text{ of } (1) \text{ satisfying (i) } u(x) \text{ has exactly } n-1 \text{ zeros in } (0, \pi); \text{ (ii) the first stationary value of } u(x) \text{ is equal to } h\Big\}.$$

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Now we pose the problem treated in this article.

Problem. For a fixed n ($n = 1, 2, \cdots$), does $\Gamma_n(g) = \Gamma_n(\tilde{g})$ imply $g = \tilde{g}$?

The purpose of this article is to give an answer to this problem for $n = 1, 2$. We begin with the first bifurcating curve. Put

$$C^1_{0,0} := \{g \in C^1(\boldsymbol{R}) \mid g(0) = g'(0) = 0\},$$

and let $g \in C^1_{0,0}$. Then, as was established in [2, Lemma 2.2], a point $(\lambda, h) \in \boldsymbol{R}^2$ belongs to the set $\Gamma_1(g)$ iff $h \neq 0$ and

$$\int_0^1 \frac{dt}{\left(\lambda(1-t^2) - \int_t^1 2h^{-1}g(hs)ds\right)^{1/2}} = \frac{\pi}{2}. \tag{3}$$

Furthermore we put

$$\hat{C}^1_{0,0} := \{\lambda(h) \in C^0(\boldsymbol{R}) \mid h\lambda'(h) \in C^0(\boldsymbol{R})\}.$$

Using the same argument as in the proof of [2, Theorem 2.3], it follows that there exists a function $\lambda(h) \in \hat{C}^1_{0,0}$ such that $\Gamma_1(g) = \{(\lambda(h), h) \mid h \in \boldsymbol{R} \setminus \{0\}\}$. The following theorem states that the nonlinearity g is unique for the first bifurcating curve.

Theorem 1. *Let $\lambda(h)$ be a function in $\hat{C}^1_{0,0}$ with $\lambda(0) = 1$. If, for $g, \tilde{g} \in C^1_{0,0}$, $\Gamma_1(g) = \Gamma_1(\tilde{g}) = \{(\lambda(h), h) \mid h \in \boldsymbol{R} \setminus \{0\}\}$, then $g = \tilde{g}$.*

Proof. Put

$$E(h, t) := \left(\lambda(h)(1-t^2) - \int_t^1 2h^{-1}g(hs)ds\right)^{1/2},$$

$$\tilde{E}(h, t) := \left(\lambda(h)(1-t^2) - \int_t^1 2h^{-1}\tilde{g}(hs)ds\right)^{1/2}.$$

Using (3) and interchanging the order of integration, we obtain, for any $h \in \boldsymbol{R}$,

$$\begin{aligned} 0 &= \int_0^1 \frac{dt}{\tilde{E}(h,t)} - \int_0^1 \frac{dt}{E(h,t)} \\ &= \int_0^1 \frac{dt}{\tilde{E}(h,t)E(h,t)(\tilde{E}(h,t)+E(h,t))} \int_t^1 2h^{-1}(\tilde{g}(hs) - g(hs))ds \\ &= \int_0^1 2h^{-1}(\tilde{g}(hs) - g(hs))ds \int_0^s \frac{dt}{\tilde{E}(h,t)E(h,t)(\tilde{E}(h,t)+E(h,t))} \\ &= h^{-2} \int_0^h 2(\tilde{g}(\xi) - g(\xi))d\xi \int_0^{\xi/h} \frac{dt}{\tilde{E}(h,t)E(h,t)(\tilde{E}(h,t)+E(h,t))}. \end{aligned}$$

Since $\tilde{g} - g$ is continuous, this implies that $\tilde{g}(h) - g(h) \equiv 0$. $\square$

We now treat the uniqueness problem for the second bifurcating curve, that is, the problem whether, for a given function $\lambda(h)$ with $\lambda(0) = 4$, a nonlinearity g satisfying

$$\Gamma_2(g) = \{(\lambda(h), h) \mid h \in \boldsymbol{R} \setminus \{0\}\} \tag{4}$$

is unique or not. In the case $\lambda(h) \equiv 4$, we have the following theorem, which gives a negative answer to the uniqueness problem.

Theorem 2 ([3, Theorem 1.1]). *There exist infinitely many* $g \in C^1_{0,0}$ *such that*

$$\Gamma_2(g) = \{(4, h) \mid h \in \boldsymbol{R} \setminus \{0\}\}.$$

However, a positive answer to the uniqueness problem is obtained under an assumption on $\lambda(h)$.

Theorem 3. *Let* $\lambda(h)$ *be a function in* $\hat{C}^1_{0,0}$ *with* $\lambda(0) = 4$ *and suppose that*

$$\lambda(h) \text{ is not constant on any interval.} \tag{5}$$

If, for $g, \tilde{g} \in C^1_{0,0}$, $\Gamma_2(g) = \Gamma_2(\tilde{g}) = \{(\lambda(h), h) \mid h \in \boldsymbol{R} \setminus \{0\}\}$, *then* $g = \tilde{g}$.

To prove Theorem 3 we need the following two lemmas.

Lemma 4. *Let* $\lambda(h) \in \hat{C}^1_{0,0}$ *and suppose that* $g(h) \in C^1_{0,0}$ *satisfies the condition* (4). *Let* $H(h)$ *be the second stationary value of a solution of* (1) *associated with* $(\lambda(h), h)$. *Then the function* $H(h)$ *has the following properties:*

(i) *For any* $h \in \boldsymbol{R}$,

$$\lambda(H(h)) = \lambda(h). \tag{6}$$

(ii) *For any* $h \in \boldsymbol{R}$,

$$\lambda(h)h^2 - \int_0^h 2g(s)ds = \lambda(h)H(h)^2 - \int_0^{H(h)} 2g(s)ds. \tag{7}$$

(iii) *For any* $h \in \boldsymbol{R}$,

$$\begin{aligned} &\int_0^1 \frac{dt}{\left(\lambda(h)(1-t^2) - \int_t^1 2h^{-1}g(hs)ds\right)^{1/2}} \\ &+ \int_0^1 \frac{dt}{\left(\lambda(h)(1-t^2) - \int_t^1 2H(h)^{-1}g(H(h)s)ds\right)^{1/2}} = \frac{\pi}{2}. \end{aligned} \tag{8}$$

(iv) $H(h)$ *is a decreasing function with* $H(0) = 0$, *of class* $C^1(\boldsymbol{R})$, *onto* $\boldsymbol{R}$.

Proof. Let $u(x)$ be a solution of (1) with $\lambda = \lambda(h)$ whose first stationary value is equal to h. Because the function $u(\pi - x)$ satisfies (1) with $\lambda = \lambda(h)$, $(\lambda(h), H(h)) \in \Gamma_2(g)$. By the assumption (4) this shows (i). An inspection of the proof of [3, Lemma 3.1] shows that the function $H(h)$ satisfies (7), (8).

Let x_0 be the point where $u(x)$ has the second stationary value $H(h)$. As is easily seen, $u''(x_0)H(h) < 0$. Hence we have $\lambda(h) > H(h)^{-1}g(H(h))$. This enables us to apply the implicit function theorem to (7) and conclude that $H(h)$ is a function of class $C^1(\boldsymbol{R})$. It is clear that $H(H(h)) = h$ for any $h \in \boldsymbol{R}$ and that $H(h) < 0$ for $h > 0$, $H(h) > 0$ for $h < 0$. This proves (iv).

Lemma 5. *Let* $\lambda(h)$ *be a continuous function on* $\boldsymbol{R}$ *satisfying the condition* (5). *If a continuous, increasing function* $K(h)$ *of* $\boldsymbol{R}$ *onto* $\boldsymbol{R}$, *with* $K(0) = 0$, *satisfies* $\lambda(K(h)) \equiv \lambda(h)$, *then* $K(h) \equiv h$.

Proof. We shall prove only that $K(h) = h$ for any $h \geqq 0$, because we can prove that $K(h) = h$ for any $h \leqq 0$ in a similar way. If the assertion were not true, then there would exist a number $h_0 \geqq 0$ such that $K(h_0) \neq h_0$. In view of the assumption $K(0) = 0$, $h_0 > 0$. We treat only the case $K(h_0) < h_0$ because the case $K(h_0) > h_0$ can be reduced to the former case if $\lambda(h)$ and $K(x)$ are replaced by $\lambda(K(h))$ and $K^{-1}(h)$ respectively.

Let $\{h_n\}_{n=1}^{\infty}$ be a sequence defined by $h_n = K(h_{n-1})$, $n = 1, 2, \cdots$. Since $K(h)$ is increasing and $h_1 = K(h_0) < h_0$, the sequence $\{h_n\}_{n=1}^{\infty}$ is decreasing, that is, $h_{n+1} < h_n$ for each $n = 0, 1, \cdots$. In view of $h_n \geqq 0$, the sequence $\{h_n\}_{n=1}^{\infty}$ converges to a limit $h_\infty \geqq 0$. By the assumption $\lambda(K(h)) \equiv \lambda(h)$ and the definition of $\{h_n\}_{n=1}^{\infty}$, we have $\lambda(h_n) = \lambda(h_0)$ for each $n = 1, 2, \cdots$. This, together with continuity of the function λ, shows that $\lambda(h_\infty) = \lambda(h_0)$. On the other hand, by the assumption on λ, there exists a number $\tilde{h}_0 \in (h_1, h_0)$ such that $\lambda(\tilde{h}_0) \neq \lambda(h_0)$. Let $\{\tilde{h}_n\}_{n=1}^{\infty}$ be a sequence defined by $\tilde{h}_n = K(\tilde{h}_{n-1})$, $n = 1, 2, \cdots$. By an induction it follows that $h_{n+1} < \tilde{h}_n < h_n$ for each $n = 0, 1, \cdots$. Hence the sequence $\{\tilde{h}_n\}_{n=1}^{\infty}$ converges to $h_\infty \geqq 0$. By the same argument as for $\{h_n\}_{n=1}^{\infty}$, we have $\lambda(h_\infty) = \lambda(\tilde{h}_0)$. This contradicts $\lambda(\tilde{h}_0) \neq \lambda(h_0)$ and completes the proof of Lemma 5.

We now give the

Proof of Theorem 3. Let $H(h)$ be the second stationary value of the solution $u(x)$ of (1) associated with $(\lambda(h), h)$, and let $\tilde{H}(h)$ be the second stationary value of the solution $\tilde{u}(x)$ of

$$\begin{cases} \dfrac{d^2\tilde{u}}{dx^2} + \lambda(h)\tilde{u} = \tilde{g}(\tilde{u}), \quad 0 < x < \pi, \\ \tilde{u}(0) = \tilde{u}(\pi) = 0 \end{cases}$$

associated with $(\lambda(h), h)$. Then, from Lemma 4, $\tilde{H}(h)$ and $H(h)$ are continuous, decreasing functions of $\boldsymbol{R}$ onto $\boldsymbol{R}$ and satisfy, for any $h \in \boldsymbol{R}$, $\lambda(\tilde{H}(h)) = \lambda(h)$ and $\lambda(H(h)) = \lambda(h)$. Hence the function $K(h) := \tilde{H}^{-1}(H(h))$ is a continuous, increasing function of $\boldsymbol{R}$ onto $\boldsymbol{R}$ and, for any $h \in \boldsymbol{R}$, $\lambda(K(h)) = \lambda(h)$. Therefore, by Lemma 5, we have $K(h) \equiv h$, and hence, $\tilde{H}(h) \equiv H(h)$.

It follows from this fact and (8) that the functions $\tilde{g}(h)$ and $g(h)$ satisfy, for any $h \in \boldsymbol{R}$,

$$\int_0^1 \frac{dt}{\tilde{E}_+(h, t)} + \int_0^1 \frac{dt}{\tilde{E}_-(h, t)} = \frac{\pi}{2}, \quad \int_0^1 \frac{dt}{E_+(h, t)} + \int_0^1 \frac{dt}{E_-(h, t)} = \frac{\pi}{2}, \tag{9}$$

where

$$\tilde{E}_+(h, t) := \left(\lambda(h)(1 - t^2) - \int_t^1 2h^{-1}\tilde{g}(hs)ds\right)^{1/2};$$

$$\tilde{E}_-(h, t) := \left(\lambda(h)(1 - t^2) - \int_t^1 2H(h)^{-1}\tilde{g}(H(h)s)ds\right)^{1/2};$$

$$E_+(h, t) := \left(\lambda(h)(1 - t^2) - \int_t^1 2h^{-1}g(hs)ds\right)^{1/2};$$

$$E_-(h, t) := \left(\lambda(h)(1 - t^2) - \int_t^1 2H(h)^{-1}g(H(h)s)ds\right)^{1/2}.$$

From (9) we have

$$\int_0^1 \frac{dt}{\tilde{E}_+(h, t)} - \int_0^1 \frac{dt}{E_+(h, t)} + \int_0^1 \frac{dt}{\tilde{E}_-(h, t)} - \int_0^1 \frac{dt}{E_-(h, t)} = 0.$$

By a way similar to that in the proof of Theorem 1 this may be rewritten as

$$\begin{aligned}&\int_0^1 h^{-1}(\tilde{g}(hs) - g(hs))ds \int_0^s \frac{\tilde{E}_+(h, t)^{-1}E_+(h, t)^{-1}}{\tilde{E}_+(h, t) + E_+(h, t)} dt \\ &\quad + \int_0^1 H(h)^{-1}(\tilde{g}(H(h)s) - g(H(h)s))ds \int_0^s \frac{\tilde{E}_-(h, t)^{-1}E_-(h, t)^{-1}}{\tilde{E}_-(h, t) + E_-(h, t)} dt = 0.\end{aligned}$$

Therefore we have, for any $h \neq 0$,

$$\begin{aligned}&\frac{1}{h^2}\int_0^h (\tilde{g}(s) - g(s))ds \int_0^{s/h} \frac{\tilde{E}_+(h, t)^{-1}E_+(h, t)^{-1}}{\tilde{E}_+(h, t) + E_+(h, t)} dt \\ &\quad + \frac{1}{H(h)^2}\int_0^h H'(s)(\tilde{g}(H(s)) - g(H(s)))ds \\ &\quad \times \int_0^{H(s)/H(h)} \frac{\tilde{E}_-(h, t)^{-1}E_-(h, t)^{-1}}{\tilde{E}_-(h, t) + E_-(h, t)} dt \\ &= 0.\end{aligned} \tag{10}$$

On the other hand, from (7), we have

$$\int_0^{H(h)} 2(\tilde{g}(\xi) - g(\xi))d\xi = \int_0^h 2(\tilde{g}(\xi) - g(\xi))d\xi.$$

This yields

$$H'(h)(\tilde{g}(H(h)) - g(H(h))) = \tilde{g}(h) - g(h). \tag{11}$$

By substituting (11) to (10) and setting

$$\begin{aligned}P(h, s) := \frac{H(h)^2}{h^2}&\int_0^{s/h} \frac{\tilde{E}_+(h, t)^{-1}E_+(h, t)^{-1}}{\tilde{E}_+(h, t) + E_+(h, t)} dt \\ &+ \int_0^{H(s)/H(h)} \frac{\tilde{E}_-(h, t)^{-1}E_-(h, t)^{-1}}{\tilde{E}_-(h, t) + E_-(h, t)} dt,\end{aligned}$$

we have, for $h \neq 0$,

$$\int_0^h P(h, s)(\tilde{g}(s) - g(s))ds = 0.$$

Since $P(h, s) > 0$, this shows that $\tilde{g}(h) - g(h) \equiv 0$. The proof of Theorem 3 is complete. □

If a continuous function $\lambda(h)$ does not satisfy the condition (5), then the function $H(h)$ is not uniquely determined by (6). It is expected that this ambiguity leads to the nonuniqueness of the nonlinearities g satisfying (4).

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