AN INVERSE PROBLEM IN BIFURCATION THEORY, III

YUTAKA KAMIMURA

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Abstract. This note is concerned with a nonlinear boundary value problem for a simple ordinary differential equation. A sufficient condition for the second bifurcating curve to determine a nonlinear term uniquely is obtained.

Consider the boundary value problem

\[ \begin{cases} \frac{d^2u}{dx^2} + \lambda u = g(u), & 0 < x < \pi, \\ u(0) = u(\pi) = 0, \end{cases} \]

where \( g \) is a real function of class \( C^1(\mathbb{R}) \) with \( g(0) = g'(0) = 0 \). By a solution of (1) we mean a pair \((\lambda, u) \in \mathbb{R} \times C^2[0, \pi]\). The assumption \( g(0) = g'(0) = 0 \) means that \( g(u) \) is smaller than the term \( \lambda u \) near \( u = 0 \), that is, \( g(u) = o(u) \) as \( u \to 0 \). Therefore the linearized problem of (1) near the trivial solution \( u \equiv 0 \) is:

\[ \begin{cases} \frac{d^2u}{dx^2} = 0, & 0 < x < \pi, \\ u(0) = u(\pi) = 0. \end{cases} \]

It is clear that the solution set of this problem consists of the solutions \((\lambda, 0)\), \( \lambda \in \mathbb{R} \), and the solutions \((n^2, h \sin nx)\) for each \( n = 1, 2, \ldots \) and any \( h \in \mathbb{R} \setminus \{0\} \). General theorems (see e.g. \([1, 5]\)) in bifurcation theory assert that the solution branches of nonlinear problems bifurcate from eigenvalues of the linearized problems. Therefore each solution branch of (1) bifurcates at the point \((n^2, 0)\). In view of this phenomenon, the solution branch of (1) emanating from \((n^2, 0)\) will henceforth be called the \( n \)th bifurcating curve and denoted by \( \Gamma_n(g) \). More precisely, the sets \( \Gamma_n(g) \), \( n = 1, 2, \ldots \), are defined by

\[ \Gamma_n(g) := \left\{ (\lambda, h) \in \mathbb{R}^2 \mid \text{there exists a solution } (\lambda, u) \in \mathbb{R} \times C^2[0, \pi] \text{ of (1)} \right\} \]

satisfying (i) \( u(x) \) has exactly \( n - 1 \) zeros in \((0, \pi)\);

(ii) the first stationary value of \( u(x) \) is equal to \( h \).

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Now we pose the problem treated in this article.

**Problem.** For a fixed \( n (n = 1, 2, \cdots) \), does \( \Gamma_n(g) = \Gamma_n(\hat{g}) \) imply \( g = \hat{g} \)?

The purpose of this article is to give an answer to this problem for \( n = 1, 2 \).

We begin with the first bifurcating curve. Put

\[
C_{0,0} := \{ g \in C^1(\mathbb{R}) \mid g(0) = g'(0) = 0 \} ,
\]

and let \( g \in C_{0,0} \). Then, as was established in [2, Lemma 2.2], a point \((\lambda, h) \in \mathbb{R}^2\) belongs to the set \( \Gamma_1(g) \) iff \( h \neq 0 \) and

\[
\int_0^1 \frac{dt}{(\lambda(1 - t^2) - \int_t^1 2h^{-1}g(s)ds)^{1/2}} = \frac{\pi}{2}.
\]

Furthermore we put

\[
\dot{C}_{0,0} := \{ \lambda(h) \in C^0(\mathbb{R}) \mid h\lambda'(h) \in C^0(\mathbb{R}) \} .
\]

Using the same argument as in the proof of [2, Theorem 2.3], it follows that there exists a function \( \lambda(h) \in \dot{C}_{0,0} \) such that \( \Gamma_1(g) = \{ (\lambda(h), h) \mid h \in \mathbb{R} \setminus \{0\} \} \). The following theorem states that the nonlinearity \( g \) is unique for the first bifurcating curve.

**Theorem 1.** Let \( \lambda(h) \) be a function in \( \dot{C}_{0,0} \) with \( \lambda(0) = 1 \). If, for \( g, \hat{g} \in C_{0,0} \),

\[
\Gamma_1(g) = \Gamma_1(\hat{g}) = \{ (\lambda(h), h) \mid h \in \mathbb{R} \setminus \{0\} \} ,
\]

then \( g = \hat{g} \).

**Proof.** Put

\[
E(h, t) := \lambda(h)(1 - t^2) - \int_t^1 2h^{-1}g(s)ds ,
\]

\[
\dot{E}(h, t) := \lambda(h)(1 - t^2) - \int_t^1 2h^{-1}\hat{g}(s)ds .
\]

Using (3) and interchanging the order of integration, we obtain, for any \( h \in \mathbb{R} \),

\[
0 = \int_0^1 \frac{dt}{\dot{E}(h, t)} - \int_0^1 \frac{dt}{E(h, t)}
= \int_0^1 \frac{dt}{\dot{E}(h, t)E(h, t) + E(h, t)} \int_t^1 2h^{-1}(\hat{g}(hs) - g(hs))ds
= \int_0^1 2h^{-1}(\hat{g}(hs) - g(hs))ds \int_0^s \frac{dt}{\dot{E}(h, t)E(h, t) + E(h, t)}
= h^{-2} \int_0^h 2(\hat{g}(\xi) - g(\xi))d\xi \int_0^{\xi/h} \frac{dt}{\dot{E}(h, t)E(h, t) + E(h, t)} .
\]

Since \( \hat{g} - g \) is continuous, this implies that \( \hat{g}(h) - g(h) \equiv 0 \). \( \square \)

We now treat the uniqueness problem for the second bifurcating curve, that is, the problem whether, for a given function \( \lambda(h) \) with \( \lambda(0) = 4 \), a nonlinearity \( g \) satisfying

\[
\Gamma_2(g) = \{ (\lambda(h), h) \mid h \in \mathbb{R} \setminus \{0\} \}
\]

is unique or not. In the case \( \lambda(h) \equiv 4 \), we have the following theorem, which gives a negative answer to the uniqueness problem.
Theorem 2 ([3, Theorem 1.1]). There exist infinitely many \( g \in C^1_{0,0} \) such that
\[
\Gamma_2(g) = \{(4, h) \mid h \in \mathbb{R} \setminus \{0\}\}.
\]

However, a positive answer to the uniqueness problem is obtained under an assumption on \( \lambda(h) \).

Theorem 3. Let \( \lambda(h) \) be a function in \( C^1_{0,0} \) with \( \lambda(0) = 4 \) and suppose that
\[
\lambda(h) \text{ is not constant on any interval.}
\]
If, for \( g, \tilde{g} \in C^1_{0,0} \), \( \Gamma_2(g) = \Gamma_2(\tilde{g}) = \{(\lambda(h), h) \mid h \in \mathbb{R} \setminus \{0\}\} \), then \( g = \tilde{g} \).

To prove Theorem 3 we need the following two lemmas.

Lemma 4. Let \( \lambda(h) \in C^1_{0,0} \) and suppose that \( g(h) \in C^1_{0,0} \) satisfies the condition
\[
\lambda(h) < \lambda(0) = 4.
\]
Let \( H(h) \) be the second stationary value of a solution of (1) associated with \( (\lambda(h), h) \). Then the function \( H(h) \) has the following properties:
(i) For any \( h \in \mathbb{R} \),
\[
\lambda(H(h)) = \lambda(h).
\]
(ii) For any \( h \in \mathbb{R} \),
\[
\lambda(h)h^2 - \int_0^h 2g(s)ds = \lambda(h)H(h)^2 - \int_0^{H(h)} 2g(s)ds.
\]
(iii) For any \( h \in \mathbb{R} \),
\[
\int_0^1 \frac{dt}{(\lambda(h)(1 - t^2) - \int_0^1 2h^{-1}g(hs)ds)^{1/2}} = \frac{\pi}{2}.
\]
(iv) \( H(h) \) is a decreasing function with \( H(0) = 0 \), of class \( C^1(\mathbb{R}) \), onto \( \mathbb{R} \).

Proof. Let \( u(x) \) be a solution of (1) with \( \lambda = \lambda(h) \) whose first stationary value is equal to \( h \). Because the function \( u(\pi - x) \) satisfies (1) with \( \lambda = \lambda(h) \), \( (\lambda(h), H(h)) \in \Gamma_2(g) \). By the assumption (4) this shows (i). An inspection of the proof of [3, Lemma 3.1] shows that the function \( H(h) \) satisfies (7), (8).

Let \( x_0 \) be the point where \( u(x) \) has the second stationary value \( H(h) \). As is easily seen, \( u''(x_0)H(h) < 0 \). Hence we have \( \lambda(h) > H(h)^{-1}g(H(h)) \). This enables us to apply the implicit function theorem to (7) and conclude that \( H(h) \) is a function of class \( C^1(\mathbb{R}) \). It is clear that \( H(H(h)) = h \) for any \( h \in \mathbb{R} \) and that \( H(h) < 0 \) for \( h > 0 \), \( H(h) > 0 \) for \( h < 0 \). This proves (iv).

Lemma 5. Let \( \lambda(h) \) be a continuous function on \( \mathbb{R} \) satisfying the condition (5).
If a continuous, increasing function \( K(h) \) of \( \mathbb{R} \) onto \( \mathbb{R} \), with \( K(0) = 0 \), satisfies \( \lambda(K(h)) \equiv \lambda(h) \), then \( K(h) \equiv h \).

Proof. We shall prove only that \( K(h) = h \) for any \( h \geq 0 \), because we can prove that \( K(h) = h \) for any \( h \leq 0 \) in a similar way. If the assertion were not true, then there would exist a number \( h_0 \geq 0 \) such that \( K(h_0) \neq h_0 \). In view of the assumption \( K(0) = 0, h_0 > 0 \). We treat only the case \( K(h_0) < h_0 \) because
the case $K(h_0) > h_0$ can be reduced to the former case if $\lambda(h)$ and $K(x)$ are replaced by $\lambda(K(h))$ and $K^{-1}(h)$ respectively.

Let $\{h_n\}_{n=1}^{\infty}$ be a sequence defined by $h_n = K(h_{n-1})$, $n = 1, 2, \ldots$. Since $K(h)$ is increasing and $h_1 = K(h_0) < h_0$, the sequence $\{h_n\}_{n=1}^{\infty}$ is decreasing, that is, $h_{n+1} < h_n$ for each $n = 0, 1, \ldots$. In view of $h_n \geq 0$, the sequence $\{h_n\}_{n=1}^{\infty}$ converges to a limit $h_\infty \geq 0$. By the assumption $\lambda(K(h)) \equiv \lambda(h)$ and the definition of $\{h_n\}_{n=1}^{\infty}$, we have $\lambda(h_n) = \lambda(h_0)$ for each $n = 1, 2, \ldots$. This, together with continuity of the function $\lambda$, shows that $\lambda(h_\infty) = \lambda(h_0)$. On the other hand, by the assumption on $\lambda$, there exists a number $\hat{h}_0 \in (h_1, h_0)$ such that $\lambda(\hat{h}_0) \neq \lambda(h_0)$. Let $\{\hat{h}_n\}_{n=1}^{\infty}$ be a sequence defined by $\hat{h}_n = K(\hat{h}_{n-1})$, $n = 1, 2, \ldots$. By an induction it follows that $h_{n+1} < \hat{h}_n < h_n$ for each $n = 0, 1, \ldots$. Hence the sequence $\{\hat{h}_n\}_{n=1}^{\infty}$ converges to $h_\infty \geq 0$. By the same argument as for $\{h_n\}_{n=1}^{\infty}$, we have $\lambda(h_\infty) = \lambda(h_0)$. This contradicts $\lambda(\hat{h}_0) \neq \lambda(h_0)$ and completes the proof of Lemma 5.

We now give the

**Proof of Theorem 3.** Let $H(h)$ be the second stationary value of the solution $u(x)$ of (1) associated with $(\lambda(h), h)$, and let $\bar{H}(h)$ be the second stationary value of the solution $\bar{u}(x)$ of

$$
\frac{d^2\bar{u}}{dx^2} + \lambda(h)\bar{u} = \tilde{g}(\bar{u}), \quad 0 < x < \pi,
\bar{u}(0) = \bar{u}(\pi) = 0
$$

associated with $(\lambda(h), h)$. Then, from Lemma 4, $\bar{H}(h)$ and $H(h)$ are continuous, decreasing functions of $\mathbf{R}$ onto $\mathbf{R}$ and satisfy, for any $h \in \mathbf{R}$, $\lambda(\bar{H}(h)) = \lambda(h)$ and $\lambda(H(h)) = \lambda(h)$. Hence the function $K(h) := \bar{H}^{-1}(H(h))$ is a continuous, increasing function of $\mathbf{R}$ onto $\mathbf{R}$ and, for any $h \in \mathbf{R}$, $\lambda(K(h)) = \lambda(h)$. Therefore, by Lemma 5, we have $K(h) \equiv h$, and hence, $\bar{H}(h) \equiv H(h)$.

It follows from this fact and (8) that the functions $\tilde{g}(h)$ and $g(h)$ satisfy, for any $h \in \mathbf{R}$,

$$
(9) \quad \int_0^1 \frac{dt}{\mathcal{E}_+(h, t)} + \int_0^1 \frac{dt}{\mathcal{E}_-(h, t)} = \frac{\pi}{2}, \quad \int_0^1 \frac{dt}{\mathcal{E}_+(h, t)} + \int_0^1 \frac{dt}{\mathcal{E}_-(h, t)} = \frac{\pi}{2},
$$

where

$$
\mathcal{E}_+(h, t) := \left( \lambda(h)(1 - t^2) - \int_t^1 2h^{-1}\tilde{g}(hs)ds \right)^{1/2};
\mathcal{E}_-(h, t) := \left( \lambda(h)(1 - t^2) - \int_t^1 2H(h)^{-1}\tilde{g}(H(h)s)ds \right)^{1/2};
\mathcal{E}_+(h, t) := \left( \lambda(h)(1 - t^2) - \int_t^1 2h^{-1}g(hs)ds \right)^{1/2};
\mathcal{E}_-(h, t) := \left( \lambda(h)(1 - t^2) - \int_t^1 2H(h)^{-1}g(H(h)s)ds \right)^{1/2}.
$$
From (9) we have

\[
\int_0^1 \frac{dt}{E_-(h, t)} - \int_0^1 \frac{dt}{E_+ (h, t)} + \int_0^1 \frac{dt}{E_-(h, t)} - \int_0^1 \frac{dt}{E_-(h, t)} = 0.
\]

By a way similar to that in the proof of Theorem 1 this may be rewritten as

\[
\int_0^1 h^{-1}(\dddot{g}(h) - g(h))ds \int_0^s \frac{\dot{E}_+(h, t)^{-1}E_+(h, t)^{-1}}{E_+(h, t) + E_+(h, t)}dt
\]

\[+ \int_0^1 H(h)^{-1}(\dddot{g}(H(h)s) - g(H(h)s))ds \int_0^s \frac{\dot{E}_-(h, t)^{-1}E_-(h, t)^{-1}}{E_-(h, t) + E_-(h, t)}dt = 0.
\]

Therefore we have, for any \( h \neq 0 \),

\[
\frac{1}{h^2} \int_0^h (\dddot{g}(s) - g(s))ds \int_0^{s/h} \frac{\dot{E}_+(h, t)^{-1}E_+(h, t)^{-1}}{E_+(h, t) + E_+(h, t)}dt
\]

\[+ \frac{1}{H(h)^2} \int_0^h H'(s)(\dddot{g}(H(s)) - g(H(s)))ds \int_0^{H(s)/H(h)} \frac{\dot{E}_-(h, t)^{-1}E_-(h, t)^{-1}}{E_-(h, t) + E_-(h, t)}dt = 0.
\]

(10)

On the other hand, from (7), we have

\[
\int_0^{H(h)} 2(\dddot{g}(\xi) - g(\xi))d\xi = \int_0^h 2(\dddot{g}(\xi) - g(\xi))d\xi.
\]

This yields

\[
H'(h)(\dddot{g}(H(h)) - g(H(h))) = \dddot{g}(h) - g(h).
\]

(11)

By substituting (11) to (10) and setting

\[
P(h, s) := \frac{H(h)^2}{h^2} \int_0^{s/h} \frac{\dot{E}_+(h, t)^{-1}E_+(h, t)^{-1}}{E_+(h, t) + E_+(h, t)}dt
\]

\[+ \int_0^{H(s)/H(h)} \frac{\dot{E}_-(h, t)^{-1}E_-(h, t)^{-1}}{E_-(h, t) + E_-(h, t)}dt,
\]

we have, for \( h \neq 0 \),

\[
\int_0^h P(h, s)(\dddot{g}(s) - g(s))ds = 0.
\]

Since \( P(h, s) > 0 \), this shows that \( \dddot{g}(h) - g(h) \equiv 0 \). The proof of Theorem 3 is complete. \( \square \)

If a continuous function \( \lambda(h) \) does not satisfy the condition (5), then the function \( H(h) \) is not uniquely determined by (6). It is expected that this ambiguity leads to the nonuniqueness of the nonlinearities \( g \) satisfying (4).
References


Department of Mathematics, Tokyo University of Fisheries, Konan 4-5-7, Minato-ku, Tokyo, 108 Japan

E-mail address: kamimura@tokyo-u-fish.ac.jp