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A COUNTEREXAMPLE ON THE SEMICONTINUITY OF MINIMA

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ABSTRACT. Let X and Y be metric spaces, Φ a multifunction from X to Y, and v a real-valued function on $X \times Y$. We give an example in which Φ is continuous, and v is continuous, inf-compact and bounded below, but the minimum function $v^*(x) := \inf_{y \in \Phi(x)} v(x, y)$ on X is not lower semicontinuous.

INTRODUCTION

An important problem in optimization, control theory and related fields is the following. Let X and Y be metric spaces, and Φ a multifunction (or set-valued map) from X to Y with $\Phi(x) \neq \phi$ for all x in X. Given a real-valued function v on the graph $\mathrm{Gr}(\Phi)$ of Φ, which is defined as

$$\mathrm{Gr}(\Phi) := \{(x, y) | x \in X, \ y \in \Phi(x)\},$$

the problem is to give conditions under which the *minimum* function

$$v^*(x) := \inf_{y \in \Phi(x)} v(x, y), \qquad x \in X,$$

is lower semicontinuous (l.s.c.).

For example, a well-known result (see, e.g., [1], p. 14, Proposition 1.7; [3], p. 148, Proposition 7.32) states that v^* is l.s.c. if $\Phi(\cdot) \equiv Y$ is compact and v is l.s.c. Other known results include Theorems 1 and 2 below.

Theorem 1 (cf. [2], p. 116, Theorem 2; [5], Theorem 10.2). *Suppose that Φ is compact-valued and upper semicontinuous. If v is l.s.c. and bounded below, then so is v^*.*

Theorem 2 (cf. [4], Lemma 3.2(f)). *Suppose that* $\mathrm{Gr}(\Phi)$ *is a Borel subset of $X \times Y$, and that v is l.s.c. and inf-compact on* $\mathrm{Gr}(\Phi)$*—where* **inf-compactness** *means that for every real number r and x in X, the set* $\{y \in \Phi(x) | v(x, y) \leq r\}$

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is compact. If, in addition, the multifunction

$$x \mapsto \Phi^*(x) := \{y \in \Phi(x) | v^*(x) = v(x, y)\}$$

is l.s.c., then v^* *is l.s.c.*

The essential difference between Theorem 1 and Theorem 2 is that the compactness and upper semicontinuity of Φ in the former is replaced by inf-compactness of v and lower semicontinuity of Φ^* in the latter. We thank the referee for pointing out that Theorem 1 also holds even if the boundedness assumption on v is dropped, which can be deduced, for instance, from Theorem 1.4.16 in J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, 1990.

The following example shows that v^* fails to be l.s.c. if the assumptions in the above theorems are weakened.

The example

Suppose that $X = \mathbb{R}$ and $Y = \Phi(x) = [0, \infty)$ for all $x \in X$, and let v be defined as

$$\begin{aligned} v(x, y) &:= 1 + y && \text{if } (x \le 0) \text{ or } (x > 0,\ 0 \le y \le 1/2x), \\ &:= (2 + 1/x) - (2x + 1)y && \text{if } x > 0,\ 1/2x < y \le 1/x, \\ &:= y - 1/x && \text{if } x > 0, y > 1/x. \end{aligned}$$

Then the multifunction $x \mapsto \Phi(x)$ is *continuous*, and v is *continuous, inf-compact* on $\mathrm{Gr}(\Phi) = \mathbb{R} \times [0, \infty)$ and bounded below (nonnegative), but

$$v^*(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0 \end{cases}$$

is *not* l.s.c.

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