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COMPACTIFICATIONS OF THE RAY WITH THE ARC AS REMAINDER ADMIT NO n-MEAN

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ABSTRACT. An n-mean on X is a function $F : X^n \to X$ which is idempotent and symmetric. In 1970 P. Bacon proved that the $\sin(1/x)$ continuum admits no 2-mean. In this paper, it is proved that if X is any metric space which contains an open line one of whose boundary components in X is an arc, then X admits no n-mean, $n \geq 2$.

1. INTRODUCTION

An n-mean $(n \geq 2)$ is a continuous function $m : X^n \to X$ satisfying the conditions:

(i) m is idempotent: $m(x, \dots, x) = x$ for each $x \in X$,

(ii) m is symmetric: $m(x_1, \dots, x_n) = m(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for each permutation σ on $\{1, \dots, n\}$.

The problem of determining those spaces which admit an n-mean was first considered by G. Auman and Caratheodory (see, e.g., [1]), and later by B. Eckmann, T. Ganea, and P. J. Hilton (see [4]). In [4] it is proved that if X is a compactum with the homotopy type of a compact polyhedron, then X admits an n-mean $(n \geq 2)$ only if X is contractible.

Means on nonlocally connected continua were first studied by A. D. Wallace [10], who conjectured that the $\sin(1/x)$ continuum admits no 2-mean. In 1968 K. Sigmon [9] showed that the $\sin(1/x)$ continuum admits no distributive mean, and in 1970 P. Bacon [3] proved Wallace's conjecture. The results that K. Sigmon and P. Bacon obtained in their deep investigation of means included some strong necessary conditions for the existence of a mean. Their approach used tools of various homology and cohomology theories. For example, K. Sigmon used Alexander Cohomology to show [8] that if X is a continuum which admits an n-mean, then $H^p(X, Z_n) = 0$, $p \geq 1$ (in Alexander Cohomology); X cannot separate $\mathbb{R}^k$, $k \geq 2$; and X is unicoherent.

A compactification of the ray $J = (0, 1]$ is a compact space X in which J is densely embedded. $X - J$ is called the remainder of the compactification.

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The purpose of this paper is to establish the following generalization of Bacon's result through essentially geometric methods.

Theorem 1.1. *If X is any compactification of the ray with the arc as remainder, then X admits no n-mean, $n \geq 2$.*

As a corollary to the above theorem we obtain the following:

Theorem 1.2. *If X is any metric space which contains an open line one of whose boundary components in X is an arc, then X admits no n-mean, $n \geq 2$.*

Notice that the $\sin(1/x)$ continuum is the simplest member within the class of compactifications of the ray with the arc as remainder. This family was shown [2] to contain uncountably many continua no one of which maps onto any other.

2. Means and essential maps

Notation and Definition 2.1. Throughout this paper the following notation will be adopted: X^n denotes the Cartesian product of n copies of X with the product topology. If $f : X \to Y$ is a map, then $f^n : X^n \to Y^n$ is defined as: $f^n(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$. π denotes the projection of $\mathbb{R}^2$ onto the y-axis. Finally, if p and q are two points of some embedding J of the ray, then $[p, q]_J$ denotes the arc in J whose endpoints are p and q. The function $f : X \to (Y, B)$, $B \subseteq Y$, is *inessential* if there exists a map $g : X \to B$ such that $g|f^{-1}(B) = f|f^{-1}(B)$. A map is *essential* if it is not inessential.

Lemma 2.2. *If X and Y are two arcs and $f : X \to Y$ is a map which maps the boundary of X onto the boundary of Y, then*

(i) *$f : X \to (Y, \operatorname{bd} Y)$ is essential, and*

(ii) *$f^n : X^n \to (Y^n, \operatorname{bd} Y^n)$ is essential.*

Proof. (i) If there were a map $g : X \to \operatorname{bd} Y$ such that $g|f^{-1}(\operatorname{bd} Y) = f$, then $g(x)$ would be a disconnected set with two points.

(ii) The map f is clearly homotopic $(\operatorname{mod} \operatorname{bd} X)$ to a homeomorphism and thus as a map of pairs $f^n : (X^n, \operatorname{bd} X^n) \to (Y^n, \operatorname{bd} Y^n)$ is homotopic to a homeomorphism $k : (X^n, \operatorname{bd} X^n) \to (Y^n, \operatorname{bd} Y^n)$. If there were a map $k : X^n \to \operatorname{bd} Y^n$ such that $k(f^n)^{-1}(\operatorname{bd} Y^n) = f^n$, then $k|\operatorname{bd} X^n$ would be homotopic to $h|\operatorname{bd} X^n$ and thus $h|\operatorname{bd} X^n \to \operatorname{bd} Y^n$ would be homotopically trivial (in $\operatorname{bd} Y^n$), which is impossible.

Note. Lemma 2.2 can be extended to the case when X is a general metric continuum, but the lemma is false if Y is allowed to be I^2. Specifically, in [7] there is constructed a compactum X with $\dim X = 2$ but $\dim X^2 = 3$. It follows from [5], Theorems VI2 and VI4 and the remarks on pages 78–79, that X has an essential mapping onto $(I^2, \operatorname{bd} I^2)$ but X^2 has no essential mapping onto $(I^4 = I^2 \times I^2, \operatorname{bd} I^4)$.

Lemma 2.3. *Let X be a continuum, and let $f : X \to (I^n, \operatorname{bd} I^n)$ be an essential map. Let K be any continuum which is the closure of an open set in I^n and which is the intersection of a sequence of open subsets of $I^n : U_1 \supset U_2 \supset U_3 \supset \cdots$, such that each $\operatorname{cl} U_i$ is homeomorphic to an n-ball. Then some continuum in X maps onto K.*

Proof. We shall show that if no component of X maps onto K, then $f : X \to (I^n, \operatorname{bd} I^n)$ is inessential. Let $C_1, C_2, \dots$ be the components of $f^{-1}(K)$. For each i pick integer $j(i)$ and open set V_i so that

(a) V_i is a component of $f^{-1}(U_{j(i)})$,

(b) $C_i \subset V_i$,

(c) $V_i \cap V_j = \varnothing$, $i \neq j$, and

(d) $f(\operatorname{cl} V_i)$ is not onto K.

We now define a new map $\hat{f} : X \to I^n$ such that $\hat{f}|X - \bigcup\{V_i\} = f|X - \bigcup\{V_i\}$ and $\hat{f}|\operatorname{cl} V_i$ is defined as follows: Let $p_i \in (K - f(\operatorname{cl} V_i)) \cap \operatorname{int} I^n$ and let r_i be a retract of $\operatorname{cl} U_{j(i)} - p_i$ onto $\operatorname{bd} V_{j(i)}$. Define $\hat{f}|\operatorname{cl} V_i = r_i \circ f|\operatorname{cl} V_i$. Note that this makes sense since $f(\operatorname{bd} V_i) \subset \operatorname{bd} V_{j(i)}$. Note also that $\hat{f}|f^{-1}(\operatorname{bd} I^n) = f|f^{-1}(\operatorname{bd} I^n)$ since $\operatorname{cl} U_{j(i)} \cap \operatorname{bd} I^2 \subset \operatorname{bd} U_{j(i)}$.

Now $\hat{f}(X) \subset I^n - (K \cap \operatorname{int} I^n)$. So pick a point $p \in K_n \cap \operatorname{int} I^n$ and let r be a retract of $I^n - \{p\}$ onto $\operatorname{bd} I^n$. Then $g = r \circ \hat{f} : X \to \operatorname{bd} I^n$ is such that $g|f^{-1}(\operatorname{bd} I^n) = f|f^{-1}(\operatorname{bd} I^n)$, which contradicts our hypotheses.

Lemma 2.4. *If G and H are open sets covering I^n such that $\{(x, x, \dots, x)|0 \leq x \leq \frac{1}{2}\} \subset G$ and $\{(x, x, \dots, x)|\frac{1}{2} \leq x \leq 1\} \subset H$, then either*

(i) *the component of G containing $(0, 0, \dots, 0)$ intersects B_1 [$\equiv$ the union of the $(n-1)$-faces of $\operatorname{bd} I^n$ which contain $(1, 1, \dots, 1)$], or*

(ii) *the component of H containing $(1, 1, \dots, 1)$ intersects B_0 [$\equiv$ the union of the $(n-1)$-faces of $\operatorname{bd} I^n$ which contain $(0, 0, \dots, 0)$].*

Proof. Let $\widetilde{G}$ [$\widetilde{H}$] be the component of G [H] containing $(0, 0, \dots, 0)$ [$(1, 1, \dots, 1)$]. $\operatorname{bd} \widetilde{G}$ separates $(0, 0, \dots, 0)$ from $(1, 1, \dots, 1)$ and thus one component D of $\operatorname{bd} \widetilde{G}$ separates I^n into open sets $U(\supset (0, 0, \dots, 0))$ and
$V(\supset (1, 1, \dots, 1))$. Note that, since $\widetilde{H} \cap G \neq \varnothing$, it must be true that $\widetilde{H} \cap D \neq \varnothing$ and thus $D \subset \widetilde{H}$. Hence either $D \cap B_0 \neq \varnothing$ or $D \cap B_0 = \varnothing$. If $D \cap B_0 \neq \varnothing$, then conclusion (ii) holds. If $D \cap B_0 = \varnothing$, then conclusion (i) holds.

Lemma 2.5. *Let $A = A_1 \cup A_2$ where A_1 and A_2 are arcs in the plane such that $A_1 \cap A_2 = \{p\}$ is an endpoint of both A_1 and A_2 and such that $\pi|A_1$ and $\pi|A_2$ are both homeomorphisms onto I. If K is any subcontinuum of I^n which intersects each $(n-1)$-face of $\operatorname{bd} I^n$ containing $(\pi(p), \pi(p), \dots, \pi(p))$, then $(\pi^n)^{-1}(K)$ is a subcontinuum of A^n.*

Proof. For each of the 2^n n-cells $A^n = A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n}$, $i_j = 1$ or 2, $\pi|A^n$ is a homeomorphism and thus $(\pi^n)^{-1}(K) \cap A^n$ is a continuum. Let

$$K_1 = (\pi^n)^{-1}(K) \cap (A_1^n) \quad \text{and} \quad K_2 = (\pi^n)^{-1}(K) \cap (A_2 \times A_1^{n-1}).$$

Then $A_1^n \cap (A_2 \times A_1^{n-1}) = \{p\} \times A_1^{n-1}$ and by hypothesis K intersects $\pi(\{p\} \times A_1^{n-1}) = \pi(p) \times A_1^{n-1}$. Thus $K_1 \cup K_2$ is connected. Continuing likewise we see that $(\pi^n)^{-1}(K)$ is connected.

3. Embedding compactifications of the ray

Theorem 3.1 [6]. *If K is a continuum which is embeddable in $\mathbb{R}^n$, then any compactification of a ray with K as remainder is embeddable in $\mathbb{R}^{n+1}$.*

The proof of the above theorem actually gives the embedding in $\mathbb{R}^n \times [0, 1]$ as the closure of the graph of a piecewise linear function $g : (0, 1] \to \mathbb{R}^n$.

In the case when K is an arc, the proof of the above theorem can be slightly modified to give the following:

Lemma 3.2. *Let S be a compactification of the ray with the arc as remainder. Then S can be embedded in the plane as the closure of the graph of a piecewise linear function $g : (0, 1] \to [0, 1]$ such that for every $y \in [0, 1]$, $f^{-1}(y)$ is a sequence in $(0, 1]$ converging to 0.*

Corollary 3.3. *Let S be a compactification of the ray with the arc as remainder. Then the following hold:*

(a) *There exists a sequence $\{q_i\}$ in $J \cap \pi^{-1}(1)$ such that $\pi|[q_{2i-1}, q_{2i}]_J$ is a surjection onto I and where $\pi^{-1}(1) \cap [q_{2i-1}, q_{2i}]_J = \{q_{2i-1}, q_{2i}\}$.*

(b) *There exists a sequence $\{p_i\}$ in $J \cap \pi^{-1}(p)$ such that $\pi|[p_{2i-1}, p_{2i}]_J$ is a surjection onto I and where $\pi^{-1}(p) \cap [p_{2i-1}, p_{2i}]_J = \{p_{2i-1}, p_{2i}\}$.*

4. Proofs of main theorems

Proof of Theorem 2.2. Let S be a compactification of the ray with the arc as remainder, and suppose that $m : S^n \to S$ is an n-mean, for some $n \geq 2$. Let $G = \{(x, y) \in S | y > \frac{1}{3}\}$, $H = \{(x, y) \in S | y < \frac{2}{3}\}$, and let $\{G', H'\}$ be an open cover of S, such that $\operatorname{cl} G' \subset G$ and $\operatorname{cl} H' \subset H$. Hence $\{m^{-1}(G'), m^{-1}(H')\}$ is an open cover of S^n. By Lemma 2.4, two cases arise:

Case (i): The component of $(1, 1, \dots, 1)$ in $m^{-1}(H') \cap I^n$ intersects B_0. Let L denote the closure of this component. By symmetry of m, it follows that L intersects each $(n-1)$-face of B_0. Let $\{q_i\}$ be the sequence guaranteed by part (a) of Lemma 3.3, and let i be chosen large enough so that $(\pi^n)^{-1}(L) \cap [q_i, q_{i+1}]_J^n \subseteq m^{-1}(H)$. Let $p \in [q_i, q_{i+1}]_J \cap \pi^{-1}(0)$.

Define a function $\pi' : [q_i, q_{i+1}]_J \to [q_i, p]_{\mathbb{R}^2} \cup [p, q_{i+1}]_{\mathbb{R}^2}$, where $A_1 = [q_i, p]_{\mathbb{R}^2}$ and $A_2 = [p, q_{i+1}]_{\mathbb{R}^2}$ denote the straight line segments joining q_i to p and q_{i+1} to p, respectively. Since $\pi|[g_i, p]_{\mathbb{R}^2}$ and $\pi|[g_{i+1}, p]_{\mathbb{R}^2}$ are both homeomorphisms onto I, it follows from Lemma 2.5 that $L' = (\pi^n)^{-1}(L) \cap (A_1 \cup A_2)^n$ is a continuum containing $(q_i, \dots, q_i)$ and $(q_{i+1}, \dots, q_{i+1})$. Since π'^n is an essential map, it follows from Lemma 2.3 that $(\pi'^n)^{-1}(L')$ contains a component L'' such that $\pi'^n(L'') = L'$. Notice that $\pi^n(L'') = \pi^n \circ \pi'^n(L'') = L$. Hence $L'' \subseteq (\pi^n)^{-1}(L) \cap [q_i, q_{i+1}]_J^n \subseteq m^{-1}(H)$. Since L'' contains the points $(q_i, \dots, q_i)$ and $(q_{i+1}, \dots, q_{i+1})$, it follows that both of them belong to the same component of $m^{-1}(H)$. This contradicts the fact that their images $m(q_i, \dots, q_i) = q_0$ and $m(q_{i+1}, \dots, q_{i+1}) = q_{i+1}$ belong to different components of H. Hence the original assumption that $m^{-1}(H) \cap I^n$ intersects B_0 is false.

Case (ii): The component of $(0, 0, \dots, 0)$ in $m^{-1}(G)$ intersects B_1. Using an argument similar to the one above but replacing the sequence $\{q_i\}$ with the sequence $\{p_i\}$ from part (b) of Lemma 3.3, we conclude that for sufficient large i, $(p_i, \dots, p_i)$ and $(p_{i+1}, \dots, p_{i+1})$ belong to the same component of $m^{-1}(G)$, contradicting the fact that their images $m(p_i, \dots, p_i) = p_i$ and $m(p_{i+1}, \dots, p_{i+1}) = p_{i+1}$ belong to different components of G. Hence the assumption that the component of $(0, \dots, 0)$ in $m^{-1}(G)$ intersects B_1 is also false. But the conclusions of the above two cases are jointly inconsistent with Lemma 2.4. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let X be a metric space which contains an open line L one of whose boundary components in X is an arc I. Then L contains a half line (ray), J, such that $\mathrm{cl}\, J - J$ is I or a subarc of I. Suppose that $m : X^n \to X$ is an n-mean. Since m is idempotent, it follows that there exists a subray $J' \subset J$ such that $\mathrm{cl}\, J' - J' = I$ and $\mathrm{cl}\, J' \subseteq m(\mathrm{cl}\, J'^n) \subseteq \mathrm{cl}\, J$. Let a, a' denote the endpoints of the rays J and J' respectively. Define a function $g : \mathrm{cl}\, J \to \mathrm{cl}\, J'$ as follows: $g|\mathrm{cl}\, J'$ is the identity and $g(x) = a'$ for each $x \in [a', a] \subset J$. Then the map $m' : \mathrm{cl}\, J'^n \to \mathrm{cl}\, J'$ given by $m' = g \circ m$ is clearly an n-mean on $\mathrm{cl}\, J'$, which contradicts Theorem 1.1, since $\mathrm{cl}\, J'$ is a compactification of the ray with the arc as remainder. This completes the proof of Theorem 1.2.

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