PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 11, November 1995

ON d-PARAMETER POINTWISE ERGODIC THEOREMS IN L_1

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $P_1, \dots, P_d$ be commuting positive linear contractions on L_1 and let $T_1, \dots, T_d$ be (not necessarily commuting) linear contractions on L_1 such that $|T_i f| \le P_i|f|$ for $1 \le i \le d$ and $f \in L_1$. In this paper we prove that if each P_i, $1 \le i \le d$, satisfies the mean ergodic theorem, then the averages $A_n(T_1, \dots, T_d)f = A_n(T_1) \cdots A_n(T_d)f$, where $A_n(T_i) = n^{-1}\sum_{k=0}^{n-1} T_i^k$, converge a.e. for every $f \in L_1$. When $T_1, \dots, T_d$ commute, we further prove that the L_1-norm convergence of the averages $A_n(P_1, \dots, P_d)f$ for every $f \in L_1$ implies the a.e. convergence of the averages $A_n(T_1, \dots, T_d)f$ for every $f \in L_1$. These improve Çömez and Lin's ergodic theorem.

1. INTRODUCTION AND THE RESULTS

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space, and let $L_p = L_p(X, \mathscr{F}, \mu)$, $1 \le p \le \infty$, denote the usual Banach spaces of real or complex functions on $(X, \mathscr{F}, \mu)$. A linear operator $T : L_p \to L_p$ is called *positive* if $f \ge 0$ implies $Tf \ge 0$, and a *contraction* if $\|T\|_p \le 1$, with $\|T\|_p$ denoting the operator norm of T on L_p. We shall say that T *satisfies the pointwise ergodic theorem* (resp. *the mean ergodic theorem*) if for any f in L_p the averages

$$A_n(T)f = \left(\frac{1}{n}\sum_{k=0}^{n-1} T^k\right) f$$

converge a.e. on X (resp. in L_p-norm).

About thirty years ago, Chacon [6] showed by a counterexample that a positive linear contraction on L_1 does not necessarily satisfy the pointwise ergodic theorem; and Ito [10] proved that if a positive linear contraction on L_1 satisfies the mean ergodic theorem, then it satisfies the pointwise ergodic theorem (cf. also [11]). Since then, generalizations and extensions of Ito's theorem have been done by several authors (see [14], [2], [13], [15] and [7]).

On the other hand, let $T_1, \dots, T_d$ be d commuting linear contractions of L_1. In 1956, Dunford and Schwartz [8] proved that if each of the operators is

Received by the editors September 1, 1993 and, in revised form, May 9, 1994; the contents of this article have been presented at the 1994 Autumn Meeting of the Mathematical Society of Japan, held at Tokyo Institute of Technology in September 1994.

1991 *Mathematics Subject Classification*. Primary 47A35.

Key words and phrases. Mean and pointwise ergodic theorems, linear contraction, linear modulus, Brunel operator.

also a contraction of L_∞, then the d-dimensional averages

$$A_n(T_1, \dots, T_d)f = A_n(T_1)\cdots A_n(T_d)f$$

converge a.e. on X. A more direct proof was given in 1973 by Brunel [3], who introduced the important tool of the auxiliary operator now called the Brunel operator. The natural question arises whether the same a.e. convergence can be established without requiring the operators to be L_∞-contractions. Assuming that each of the operators is positive, McGrath [13] showed that it is enough that for some $p > 1$ we have that each operator is a contraction of L_p; and one of the authors proved in [15] that it is also enough that each operator satisfies the mean ergodic theorem. Without assuming the positivity of the operators, Çömez and Lin [7] have recently proved that if the linear moduli τ_i of the operators T_i commute and each τ_i satisfies the mean ergodic theorem, then the a.e. convergence of the d-dimensional averages holds.

The purpose of this paper is to generalize and improve these results as follows.

Theorem 1. *Let $P_1, \dots, P_d$ be commuting positive linear contractions on L_1 and let $T_1, \dots, T_d$ be commuting linear contractions on L_1 such that $|T_i f| \le P_i|f|$ for $1 \le i \le d$ and $f \in L_1$. Then we have:*

(a) *For every $f \in L_1$ the averages $A_n(T_1, \dots, T_d)f$ converge a.e. on X, when the averages $A_n(P_1, \dots, P_d)f$ converge a.e. on X for all $f \in L_1$.*

(b) *The averages $A_n(P_1, \dots, P_d)f$ converge a.e. on X for all $f \in L_1$, when the Brunel operator U corresponding to $P_1, \dots, P_d$ satisfies the pointwise ergodic theorem.*

Corollary. *Let $P_1, \dots, P_d$ be commuting positive linear contractions of L_1 and let $T_1, \dots, T_d$ be commuting linear contractions of L_1 such that $|T_i f| \le P_i|f|$ for $1 \le i \le d$ and $f \in L_1$. Then we have:*

(a) *For every $f \in L_1$ the averages $A_n(T_1, \dots, T_d)f$ converge a.e. on X, when the averages $A_n(P_1, \dots, P_d)f$ converge in L_1-norm for all $f \in L_1$.*

(b) *For every $f \in L_1$ the averages $A_n(T_1, \dots, T_d)f$ converge a.e. on X, when the Brunel operator U corresponding to $P_1, \dots, P_d$ satisfies either $\|U\|_p \le 1$ for some $p > 1$ or $\sup_k \|U^k\|_\infty < \infty$.*

By an easy induction argument we see that if $P_1, \dots, P_d$ are (not necessarily commuting) positive linear contractions of L_1 which satisfy the mean ergodic theorem, then for every $f \in L_1$ the averages $A_n(P_1, \dots, P_d)f$ converge in L_1-norm. However, the converse does not hold in general. To see this, let P_1 be any positive linear contraction of L_1 which does not satisfy the mean ergodic theorem and let $P_2 = 0$. Then, clearly, we have $\lim_n \|A_n(P_1, P_2)f\|_1 = 0$ for all $f \in L_1$.

Theorem 2. *Let $P_1, \dots, P_d$ be commuting positive linear contractions of L_1 which satisfy the mean ergodic theorem, and let $T_1, \dots, T_d$ be (not necessarily commuting) linear contractions of L_1 such that $|T_i f| \le P_i|f|$ for $1 \le i \le d$ and $f \in L_1$. Then for every $f \in L_1$ the averages $A_n(T_1, \dots, T_d)f$ converge a.e. on X.*

Theorem 3 (cf. [7], Theorem 2.4). *Let $P_1, \dots, P_d$ be commuting positive linear contractions of L_1 and let $T_1, \dots, T_d$ be (not necessarily commuting) linear contractions of L_1 such that $|T_i f| \le P_i|f|$ for $1 \le i \le d$ and $f \in L_1$. If the*

Brunel operator U corresponding to $P_1, \ldots, P_d$ satisfies $\|U\|_\infty \le 1$, then for every $f \in L_1$ the averages $A_n(T_1, \ldots, T_d)f$ converge a.e. on X.

2. Proofs

Lemma. *Let $P_1, \ldots, P_d$ be commuting positive linear contractions of L_1 and let $T_1, \ldots, T_d$ be commuting linear contractions of L_1 such that $|T_i f| \le P_i|f|$ for $1 \le i \le d$ and $f \in L_1$. If u is a (not necessarily integrable) nonnegative function on $(X, \mathscr{F}, \mu)$ such that $P_i u \le u < \infty$ a.e. on X for every $1 \le i \le d$, then the averages $A_n(T_1, \ldots, T_d)f$ converge a.e. on the set $Y = \{x : u(x) > 0\}$ for all $f \in L_1$.*

Proof. When $d = 1$, this follows from the Chacon general ratio ergodic theorem (see e.g. [12], Theorem 4.1.11, p. 164). We then use an induction argument. Let $0 \le f \in L_1$ and $k = 0, 1, \ldots$ be fixed arbitrarily. Since by commutativity of the operators P_i we obtain

$$(1) \qquad n^{-d} \sum_{0 \le n_2, \ldots, n_d < n} P_1^k P_2^{n_2} \cdots P_d^{n_d} f = \frac{1}{n} A_n(P_2, \ldots, P_d)(P_1^k f),$$

and since by the induction hypothesis applied to $P_2, \ldots, P_d$ the averages

$$A_n(P_2, \ldots, P_d)(P_1^k f)$$

converge a.e. on Y, it follows that

$$(2) \qquad \lim_n n^{-d} \sum_{0 \le n_2, \ldots, n_d < n} P_1^k P_2^{n_2} \cdots P_d^{n_d} f = 0 \quad \text{a.e. on } Y.$$

Applying the same argument to $P_2^k, \ldots, P_d^k$ and using the commutativity of $T_1, \ldots, T_d$ we see that for each $N \ge 1$ if we set

$$V = T_1 T_2 \cdots T_d,$$

then

$$(3) \qquad \begin{aligned} &\left| A_n(T_1, \ldots, T_d)f - \left(1 - \frac{N}{n}\right)^d A_{n-N}(T_1, \ldots, T_d)(V^N f) \right| \\ &\le \sum_{j=1}^{d} \sum_{k=0}^{N-1} \frac{1}{n^d} \left[\sum_{i \ne j,\ n_i < n} P_1^{n_1} \cdots P_{j-1}^{n_{j-1}} P_{j+1}^{n_{j+1}} \cdots P_d^{n_d} (P_j^k f) \right] \\ &\to 0 \quad \text{a.e. on } Y \text{ as } n \to \infty. \end{aligned}$$

Put

$$(4) \qquad f_N = 1_Y \cdot (V^N f) \text{ and } g_N = (V^N f) - f_N,$$

where 1_Y denotes the indicator function of Y. Since $f_N \in L_1(Y)$ and $P_i u \le u < \infty$ for each $1 \le i \le d$, changing the measure μ to $u d\mu$ and applying the Dunford and Schwartz ergodic theorem (see e.g. [12], Theorem 6.3.5, p. 215) we see that

$$(5) \qquad \lim_n A_n(T_1, \ldots, T_d) f_N \text{ exists a.e. on } Y.$$

(Since f_N is supported on Y, the support of the subinvariant measure, and supports of subinvariant measures are absorbing sets, we have that the convergence in (5) actually holds at almost all points of X; this was noticed by the referee.) Therefore the function

$$F(x) = \limsup_{m,n\to\infty} |A_m(T_1, \dots, T_d)f(x) - A_n(T_1, \dots, T_d)f(x)| \tag{6}$$

satisfies

$$\begin{aligned} F(x) &\le \limsup_{m,n\to\infty} |A_m(T_1, \dots, T_d)g_N(x) - A_n(T_1, \dots, T_d)g_N(x)| \\ &\le 2\limsup_n |A_n(T_1, \dots, T_d)g_N(x)| \quad \text{a.e. on } Y. \end{aligned}$$

Then, putting $h_N = (P_1 \cdots P_d)^N f - 1_Y \cdot (P_1 \cdots P_d)^N f$, we have

$$\begin{cases} |g_N| \le h_N \in L_1(X\backslash Y), \\ F \le 2\limsup_n A_n(P_1, \dots, P_d)h_N \quad \text{a.e. on } Y. \end{cases} \tag{7}$$

Since $P_i L_1(Y) \subset L_1(Y)$ for each $1 \le i \le d$, it follows that

$$\begin{aligned} h_{N+n} &= 1_{X-Y} \cdot (P_1 \cdots P_d)^{N+n} f \\ &= 1_{X-Y} \cdot (P_1 \cdots P_d)^n (h_N + 1_Y (P_1 \cdots P_d)^N f) \\ &= 1_{X-Y} \cdot (P_1 \cdots P_d)^n h_N. \end{aligned}$$

In particular, this implies $\|h_N\|_1 \ge \|h_{N+1}\|_1$. Hence we can put

$$\alpha = \lim_N \|h_N\|_1.$$

Then, given an $\varepsilon > 0$, we can choose $N \ge 1$ so that

$$\|h_N\|_1 < \alpha + \varepsilon.$$

Since $P_j L_1(Y) \subset L_1(Y)$, we have $1_{X-Y} \cdot P_j f = 1_{X-Y} \cdot P_j(1_{X-Y} f)$; it follows that

$$\|1_{X-Y} \cdot P_j f\|_1 \le \|1_{X-Y} f\|_1.$$

Thus for any $0 \le n_1, n_2, \dots, n_d < n$ we have

$$\begin{aligned} \alpha + \varepsilon > \|h_N\|_1 &\ge \|1_{X-Y} \cdot (P_1^{n_1} \cdots p_d^{n_d} h_N)\|_1 \\ &\ge \|1_{X-Y} \cdot (P_1 \cdots P_d)^n h_N\|_1 = \|h_{N+n}\|_1 \ge \alpha, \end{aligned}$$

which implies that

$$\begin{aligned} \|1_Y \cdot (P_1^{n_1} \cdots P_d^{n_d} h_N)\|_1 &= \|P_1^{n_1} \cdots P_d^{n_d} h_N\|_1 - \|1_{X-Y} \cdot (P_1^{n_1} \cdots P_d^{n_d} h_N)\|_1 \\ &\le \|h_N\|_1 - \alpha < \varepsilon. \end{aligned} \tag{8}$$

We now consider the Brunel operator U corresponding to $P_1, \dots, P_d$. It follows (cf. [12], Theorem 6.3.4, p. 213) that for all $0 \le h \in L_1$

$$\limsup_n A_n(P_1, \dots, P_d)h \le C_d \cdot \limsup_n A_n(U)h \quad \text{on } X, \tag{9}$$

where $C_d > 0$ is an absolute constant depending only on d. Since U has the form

$$U = \sum_{n_1, \dots, n_d \ge 0} a(n_1, \dots, n_d) P_1^{n_1} \cdots P_d^{n_d}, \tag{10}$$

with

$$a(n_1, \dots, n_d) > 0 \quad \text{and} \quad \sum_{n_1, \dots, n_d \geq 0} a(n_1, \dots, n_d) = 1, \tag{11}$$

it follows from (8) that $\int_Y U^k h_N \, d\mu < \varepsilon$ for all $k \geq 0$.

On the other hand, by the fact that $Uu \leq u$ a.e. on X which follows from $P_i u \leq u$ for each $1 \leq i \leq d$, we may apply the Chacon general ratio ergodic theorem to infer that for every $0 \leq h \in L_1$ the averages $A_n(U)h$ converge a.e. on Y. Consequently, using (7), (9), and Fatou's lemma,

$$\begin{aligned}\int_Y F \, d\mu &\leq 2C_d \cdot \int_Y \lim_n A_n(U) h_N \, d\mu \\ &\leq 2C_d \liminf_n \int_Y A_n(U) h_N \, d\mu \leq 2C_d \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows that $F(x) = 0$ a.e. on Y, completing the proof.

Proof of Theorem 1. To prove (a), let $0 \leq f \in L_1$ be fixed arbitrarily and write, using the hypothesis,

$$f^\sim(x) = \lim_n A_n(P_1, \dots, P_d) f(x) \quad \text{a.e. on } X.$$

(Here we may assume without loss of generality that $d \geq 2$. For, if necessary, add the identity operator.) It is easily seen that $P_i f^\sim \leq f^\sim$ for each $1 \leq i \leq d$. Thus the Brunel operator U corresponding to $P_1, \dots, P_d$ satisfies

$$0 \leq U f^\sim \leq f^\sim \in L_1.$$

On the other hand denoting by C the conservative part of U, $0 \leq f \in L_1$ implies

$$n^{-1} \sum_{k=0}^{n-1} U^k f \to 0 \quad \text{a.e. on } X \backslash C,$$

so we obtain from inequality (9) applied to f instead of h that

$$\{x : f^\sim(x) > 0\} \subset C;$$

therefore $U f^\sim \in L_1(C)$ and

$$\|U f^\sim\|_1 = \int_C U f^\sim \, d\mu = \int f^\sim (U^* 1_C) \, d\mu = \int_C f^\sim \, d\mu = \int f^\sim \, d\mu = \|f^\sim\|_1.$$

It follows that $U f^\sim = f^\sim$, and by the Brunel-Falkowitz Lemma (see [12], p. 82) $P_i f^\sim = f^\sim$ for each $1 \leq i \leq d$. Hence we may apply our Lemma to infer that the averages $A_n(T_1, \dots, T_d) f$ converge a.e. on the set $\{x : f^\sim(x) > 0\}$.

On the other hand, it is clear that

$$\lim_n |A_n(T_1, \dots, T_d) f| \leq \lim_n A_n(P_1, \dots, P_d) f = 0$$

a.e. on $\{x : f^\sim(x) = 0\}$.

To prove (b), let $0 \leq f \in L_1$ be fixed and write

$$f^*(x) = \lim_n A_n(U) f(x) \quad \text{a.e. on } X.$$

Since $0 \le Uf^* = f^* \in L_1$, it follows that $P_i f^* = f^*$ for each $1 \le i \le d$. Our Lemma yields the a.e. convergence of $A_n(P_1, \dots, P_d)f$ on $\{x : f^*(x) > 0\}$, and inequality (9) yields convergence (to 0) a.e. on $\{x : f^*(x) = 0\}$.

Proof of the Corollary. (a) Çömez and Lin [7] proved that if for every $f \in L_1$ the averages $A_n(P_1, \dots, P_d)f$ converge in L_1-norm, then the Brunel operator U satisfies the mean ergodic theorem. Thus U satisfies the pointwise ergodic theorem, by Ito's theorem. And (a) follows from Theorem 1.

(b) Since U is a positive linear contraction of L_1 which satisfies either $\|U\|_p \le 1$ or $\sup_k \|U^k\|_\infty < \infty$, U also satisfies the pointwise ergodic theorem in L_1 by Akcoglu and Chacon's ergodic theorem [2] or by Hasegawa, Sato, and Tsurumi's ergodic theorem [9]. Thus (b) follows from Theorem 1, too.

Proof of Theorem 2. When $d = 1$, this is a special case of (a) of the corollary. We then use an induction argument. Since the Brunel operator U corresponding to the commuting operators $P_1, \dots, P_d$ satisfies the mean ergodic theorem by Çömez and Lin [7] (cf. also the proof of the lemma in [15], where this result is proved explicitly using a different method), U satisfies the pointwise ergodic theorem, by Ito's theorem. This and the obvious inequality

$$|A_n(T_1, \dots, T_d)f| \le A_n(P_1, \dots, P_d)|f| \tag{12}$$

together with inequality (9) imply that for all $f \in L_1$

$$\sup_n |A_n(T_1, \dots, T_d)f| < \infty \quad \text{a.e. on } X. \tag{13}$$

Hence Banach's convergence principle (cf. e.g. [12], Theorem 1.7.2, p. 64) completes the proof, when we see that the set

$$\left\{ f \in L_1 : \lim_n A_n(T_1, \dots, T_d)f \text{ exists a.e. on } X \right\}$$

is dense in L_1. To see this we note that, since P_d satisfies the mean ergodic theorem, so does T_d (cf. [7]), and thus the set

$$\{g + (f - T_d f) : T_d g = g \text{ and } f \in L_1\}$$

is dense in L_1.

For the function $h = g + (f - T_d f)$, with $T_d g = g$ and $f \in L_1$, we have

$$A_n(T_1, \dots, T_d)h = A_n(T_1, \dots, T_{d-1})g + \frac{1}{n} A_n(T_1, \dots, T_{d-1})(f - T_d^n f).$$

The induction hypothesis implies that

$$\lim_n A_n(T_1, \dots, T_{d-1})g \text{ exists a.e. on } X.$$

Similarly,

$$\lim_n \frac{1}{n} A_n(T_1, \dots, T_{d-1})f = 0 \quad \text{a.e. on } X.$$

On the other hand, since $P_1, \dots, P_d$ commute and

$$\|A_n(P_1, \dots, P_d)(|f| - P_d|f|)\|_1 \to 0 \quad \text{as } n \to \infty,$$

we can apply (a) of the Corollary to $P_1, \ldots, P_d$ instead of $T_1, \ldots, T_d$ to obtain that

$$\begin{aligned}\lim_n \frac{1}{n} A_n(P_1, \ldots, P_{d-1})(|f| - P_d^n|f|) \\ = \lim_n A_n(P_1, \ldots, P_{d-1}, P_d)(|f| - P_d|f|) \\ = 0 \quad \text{a.e. on } X.\end{aligned}$$

This can be applied to infer that

$$\lim_n \left| \frac{1}{n} A_n(T_1, \ldots, T_{d-1}) T_d^n f \right| \leq \lim_n \frac{1}{n} A_n(P_1, \ldots, P_{d-1}) P_d^n |f| = 0$$

a.e. on X, completing the proof.

Proof of Theorem 3. Since U satisfies the pointwise ergodic theorem, write for any $0 \leq f \in L_1$

$$f^*(x) = \lim_n A_n(U) f(x) \quad \text{a.e. on } X.$$

We then have $0 \leq Uf^* = f^* \in L_1$. Since $P_1, \ldots, P_d$ commute, it follows that $P_i f^* = f^*$ for each $1 \leq i \leq d$. Further, if we set $Y = \{x : f^*(x) > 0\}$ and $Z = \{x : f^*(x) = 0\}$, then, by the fact that $\|U\|_\infty \leq 1$, it follows that

$$UL_1(Y) \subset L_1(Y) \text{ and } UL_1(Z) \subset L_1(Z).$$

(In fact, this can be seen as follows: Let C and D be the conservative and dissipative parts of U. For any positive L_1-contraction, C is absorbing (cf. [12], p. 118). Since $\|U\|_\infty \leq 1$, we can apply Feldman's result (cf. [12], p. 131) to obtain that also D is absorbing for U. Since Y is the support of a finite invariant measure, it is absorbing, and $Y \subset C$. Hence $C \backslash Y$ is absorbing, and so is $Z = (C \backslash Y) \cup D$.) It follows that $P_i L_1(Y) \subset L_1(Y)$ and $P_i L_1(Z) \subset L_1(Z)$ for each $1 \leq i \leq d$.

On the other hand, by (9) it is clear that

$$\lim_n |A_n(T_1, \ldots, T_d) f| \leq \lim_n A_n(P_1, \ldots, P_d) f \leq C_d f^* = 0 \quad \text{a.e. on } Z$$

and

$$\begin{aligned}\lim_n |A_n(T_1, \ldots, T_d)(1_Z f)| \leq \lim_n A_n(P_1, \ldots, P_d)(1_Z f) \\ \leq C_d \cdot \lim_n A_n(U)(1_Z f) \leq C_d \cdot 1_Z f^* = 0 \quad \text{a.e. on } X.\end{aligned}$$

Therefore in order to prove the a.e. convergence of $A_n(T_1, \ldots, T_d) f$, we may assume without loss of generality that $X = Y$. Then, since each P_i $(1 \leq i \leq d)$ satisfies the mean ergodic theorem, Theorem 2 establishes the desired conclusion.

3. Concluding remarks

In Theorem 3, if we further assume that $T_1, \ldots, T_d$ commute, then the conclusion of Theorem 3 remains true even if the condition $\|U\|_\infty \leq 1$ is replaced by the weaker condition $\sup_k \|U^k\|_\infty < \infty$ or the condition $\|U\|_p \leq 1$ for some $1 < p < \infty$. (See (b) of the corollary.) On the other hand, Theorem 2.4 in [7] states that if $T_1, \ldots, T_d$ are commuting linear contractions of L_1

such that the Brunel operator U corresponding to $T_1, \dots, T_d$ is also an L_∞-contraction, then for every $f \in L_1$ the averages $A_n(T_1, \dots, T_d)f$ converge a.e. on X. However it seems to the authors that this theorem has not been proved; only the a.e. finiteness of $\sup_n |A_n(T_1, \dots, T_d)f|$ is shown in [7], but convergence on a dense subset of L_1 is still missing. The authors learned from Michael Lin (a private communication, July 1993) that Theorem 2.4 can be proved if in addition to the commutativity of $T_1, \dots, T_d$ the commutativity of their moduli $\tau_1, \dots, \tau_d$ is also assumed. We note that Theorem 3 improves this result by showing that the commutativity of the moduli alone is sufficient. We also note (cf. Proposition in [15]) that the conclusion of Theorem 2.4 remains true without assuming the commutativity of $\tau_1, \dots, \tau_d$ if for some $p > 1$ we have $\|\tau_i\|_p \le 1$ for each $1 \le i \le d$. See also [1] and [13]. We finally prove the following extension of the Dunford-Schwartz ergodic theorem.

Theorem 4. *Let $P_1, \dots, P_d$ be positive linear contractions of L_1 such that*

$$\sup\{\|A_n(P_i)\|_\infty : n \ge 1,\ 1 \le i \le d\} = K < \infty, \tag{14}$$

where $A_n(P_i) = n^{-1}\sum_{k=0}^{n-1} P_i^k$, and let $T_1, \dots, T_d$ be linear contractions of L_1 such that $|T_i f| \le P_i|f|$ for $1 \le i \le d$ and $f \in L_1$. If either the operators $P_1, \dots, P_d$ or the operators $T_1, \dots, T_d$ commute, then for every $f \in L_1$ the averages $A_n(T_1, \dots, T_d)f$ converge a.e. on X.

Proof. We first consider the case $d = 1$. For an $f \in L_p$ with $1 \le p < \infty$, let

$$f^\sharp = \sup_n A_n(P_1)|f| \quad \text{a.e. on } X, \tag{15}$$

and if a is a positive real number, denote

$$f^{a^-}(x) = [\operatorname{sgn} f(x)] \cdot \min(a, |f(x)|)$$

and

$$f^{a^+}(x) = f(x) - f^{a^-}(x).$$

Since the positive linear contraction P_1 of L_1 satisfies $\sup_n \|A_n(P_1)\|_\infty \le K$ by hypothesis, the proof of Chacon's maximal ergodic theorem [5] can be easily modified to yield that

$$\int_{\{x : f^\sharp(x) > K^2 a\}} (a - |f^{a^-}|)\, d\mu \le \int |f^{a^+}|\, d\mu, \tag{16}$$

where, if necessary, we may suppose that $\|P_1\|_\infty < K$. Thus it follows from a standard argument (see e.g. the proof of Theorem 3 in [9]) that

$$f^\sharp < \infty \quad \text{a.e. on } X \tag{17}$$

and that if, in particular, $1 < p < \infty$, then

$$\|f^\sharp\|_p \le \mathscr{K}_p \|f\|_p, \tag{18}$$

where $\mathscr{K}_p$ is an absolute constant depending only on p and K.

If $f \in L_1$, then, choosing a strictly positive function $g \in L_1 \cap L_\infty$, we have that

$$\begin{aligned} \frac{1}{n}|P_1^n f| &\le |A_n(P_1)g| \frac{P_1^n|f|}{\sum_{k=0}^{n-1} P_1^k g} \\ &\le K\|g\|_\infty \frac{P_1^n|f|}{\sum_{k=0}^{n-1} P_1^k g} \to 0 \quad \text{a.e. on } X \end{aligned} \tag{19}$$

by the Chacon-Ornstein lemma (see e.g. [12], Lemma 3.2.3, p. 121).

Thus if $1 < p < \infty$, then, since the set

$$\{g + (h - P_1 h) : P_1 g = g \in L_p,\ h \in L_p \cap L_1\}$$

is a dense subspace of L_p by the mean ergodic theorem for P_1 on L_p, Banach's convergence principle, (17) and (19) imply that the averages $A_n(P_1)f$ converge a.e. on X for all $f \in L_p$. Using this and the fact that $L_1 \cap L_p$ is a dense subspace of L_1, we then see that the averages $A_n(P_1)f$ converge a.e. on X for all $f \in L_1$. It is now immediate to see that $\lim_n A_n(T_1)f$ exists a.e. on X for all $f \in L_p$ with $1 \leq p < \infty$.

Since the case $d = 1$ has been done, we now proceed by an induction argument. First we define the operators U_i, $1 \leq i \leq d$, by the relation

$$U_i = \sum_{k=0}^{\infty} a_k P_i^k, \tag{20}$$

where the a_k are the coefficients in the expansion

$$x^{-1}(1 - \sqrt{1 - x}) = \sum_{k=0}^{\infty} a_k x^k \qquad (x \neq 0,\ |x| \leq 1). \tag{21}$$

It follows from [4] that the operators U_i are positive linear contractions of L_1 such that

$$\sup\{\|U_i^k\|_\infty : k \geq 0,\ 1 \leq i \leq d\} = K^\sim < \infty. \tag{22}$$

Here we remember that each U_i may be regarded as the Brunel operator corresponding to the operator P_i and the identity operator I.

Put $V_{2i} = P_i$ and $V_{2i+1} = I$ for $1 \leq i \leq d$, and define the operator W by the relation

$$W = \sum_{n_1, \dots, n_{2d} \geq 0} a(n_1, \dots, n_{2d}) V_1^{n_1} V_2^{n_2} \cdots V_{2d}^{n_{2d}}, \tag{23}$$

where the coefficients $a(n_1, \dots, n_{2d})$ are those used in the definition of the Brunel operator corresponding to $2d$ contraction operators of L_1 (see e.g. [12], Theorem 6.3.4, p. 213). It follows from the argument of Brunel that W has the form

$$W = \sum_{n_1, \dots, n_d \geq 0} b(n_1, \dots, n_d) U_1^{n_1} \cdots U_d^{n_d} \tag{24}$$

with

$$b(n_1, \dots, n_d) > 0 \quad \text{and} \quad \sum_{n_1, \dots, n_d \geq 0} b(n_1, \dots, n_d) = 1. \tag{25}$$

Hence by (22) W is a positive linear contraction of L_1 such that

$$\sup_k \|W^k\|_\infty \leq K^\sim < \infty.$$

Further, since either the operators $P_1, \dots, P_d$ or the operators $T_1, \dots, T_d$ commute by hypothesis, it follows that for any $f \in L_p$ with $1 \leq p < \infty$,

$$\sup_n |A_n(T_1, \dots, T_d)f| \leq C_{2d} \cdot \sup_n A_n(W)|f| \quad \text{a.e. on } X; \tag{26}$$

and since the averages $A_n(W)|f|$ converge a.e. on X, we see that

$$\sup_n |A_n(T_1, \dots, T_d)f| < \infty \quad \text{a.e. on } X. \tag{27}$$

Let us fix p with $1 < p < \infty$. Since $L_p \cap L_1$ is a dense subspace of L_1 and since the set

$$\{g + (f - T_d f) : T_d g = g \in L_p, f \in L_p \cap L_1\}$$

is a dense subspace of L_p by the mean ergodic theorem for T_d on L_p, Banach's convergence principle together with inequality (27) implies that for the proof of the theorem it is sufficient to check the a.e. convergence of $A_n(T_1, \dots, T_d)h$, where $h = g + (f - T_d f)$ with $T_d g = g \in L_p$ and $f \in L_p \cap L_1$. As

$$A_n(T_1, \dots, T_d)h = A_n(T_1, \dots, T_{d-1})g + \frac{1}{n}A_n(T_1, \dots, T_{d-1})(f - T_d^n f),$$

and by the induction hypothesis the limit

$$\lim_n A_n(T_1, \dots, T_{d-1})g$$

exists a.e. on X, it is then sufficient to check the a.e. convergence of

$$\frac{1}{n}A_n(T_1, \dots, T_{d-1})(f - T_d^n f).$$

To do this, let for each $N \geq 1$,

$$h_N = \sup_{n \geq N} \frac{1}{n}|f - T_d^n f| = \sup_{n \geq N} |A_n(T_d)(f - T_d f)|. \tag{28}$$

By using inequality (18) applied to P_d and $|f - T_d f|$ instead of P_1 and f and by using the theorem for $d = 1$ we observe that

$$h_1 \in L_p^+ \quad \text{and} \quad h_N \downarrow 0 \quad \text{a.e. on } X. \tag{29}$$

It also follows, as before (cf. inequality (26)), that

$$\begin{aligned} &\limsup_n \left| \frac{1}{n}A_n(T_1, \dots, T_{d-1})(f - T_d^n f) \right| \\ &\qquad \leq C_{2(d-1)} \cdot \sup_n A_n(W_{d-1})h_N \quad \text{a.e. on } X, \end{aligned} \tag{30}$$

where the operator W_{d-1} is defined using the operators $P_1, \dots, P_{d-1}$ as in (23), thus it is a positive linear contraction of L_1 such that

$$\sup_k \|W_{d-1}^k\|_\infty \leq K^\sim < \infty.$$

By (29) together with inequality (18) applied to W_{d-1} and h_N instead of P_1 and f we see that the functions

$$h_N^\sharp = \sup_n A_n(W_{d-1})h_N \qquad (N = 1, 2, \dots)$$

satisfy $h_1^\sharp \in L_p^+$ and $h_N^\sharp \downarrow 0$ a.e. on X. Consequently by (30) we get

$$\lim_n \frac{1}{n}A_n(T_1, \dots, T_{d-1})(f - T_d^n f) = 0 \quad \text{a.e. on } X.$$

This completes the proof.

Remark. By virtue of Theorem 4 for $d = 1$ we see that the condition

$$\sup_k \|U^k\|_\infty < \infty$$

in (b) of the Corollary to Theorem 1 can be replaced by the weaker condition $\sup_n \|A_n(U)\|_\infty < \infty$.

Acknowledgment

In conclusion, the authors would like to express their gratitude to the referee for helpful comments which made the paper readable.

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