PROJECTIONS IN SOME SIMPLE C*-CROSSED PRODUCTS

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Abstract. Let \( \alpha \) be an outer action by a finite group \( G \) on a simple C*-algebra \( A \). If each hereditary C*-subalgebra of \( A \) has an approximate identity consisting of projections, then every hereditary C*-subalgebra of the crossed product \( A \times_\alpha G \) has a projection.

1. Introduction

A C*-algebra \( A \) is said to have FS if the set of all selfadjoint elements with finite spectra (as an element of \( \hat{A} \), the unital C*-algebra obtained by adjoining the unit to \( A \)) is norm dense in \( A_{sa} \), equally, if every hereditary C*-subalgebra of \( A \) has an approximate identity consisting of projections [4]. This class of C*-algebras includes AF-algebras, von Neumann algebras [4], purely infinite simple C*-algebras [10], etc. A purely infinite C*-algebra is a C*-algebra such that any of its hereditary C*-subalgebra is infinite, that is, has an infinite projection. If each hereditary C*-subalgebra of a C*-algebra \( A \) has a non-zero projection, then we say that \( A \) has SP. There are many examples of C*-algebras which do not have FS but SP. For example, consider a purely infinite C*-algebra \( C \otimes \mathcal{K} \), where \( \mathcal{K} \) is the algebra of compact operators on a separable infinite-dimensional Hilbert space \( \mathcal{H} \) and \( C \) is the Calkin algebra \( B(\mathcal{H})/\mathcal{K} \). Then its multiplier algebra \( M(C \otimes \mathcal{K}) \) is also a purely infinite C*-algebra which does not have FS [11] but has SP, since the multiplier algebra of a simple (or primitive, in general) C*-algebra with SP obviously has SP.

While there are various examples of projectionless simple C*-algebras, a large class of simple C*-algebras are also known to contain projections [2].

In [3] Blackadar and Kumjian construct a simple C*-algebra which does not have FS but SP. We do not know whether the two conditions FS and SP are equivalent or not for infinite simple C*-algebras. Both conditions mean that a C*-algebra abounds in its projections so that if \( A \) is an infinite-dimensional simple C*-algebra with SP (or FS), then \( A \) contains no minimal projections.

We show in this short note that the crossed product \( A \times_\alpha G \) by an outer action \( \alpha \) of a finite group \( G \) has SP whenever \( A \) is a simple C*-algebra with FS. In [6,
Example 9, Elliott showed that there is an action of \( \mathbb{Z}_2 \) on a simple \( C^* \)-algebra which does not have FS but the crossed product does. So by Takesaki-Takai duality, it follows that the property FS is not necessarily preserved in crossed products by finite groups.

2. Projections in simple \( C^* \)-crossed products

Throughout this paper \( A_z \) denotes the hereditary \( C^* \)-subalgebra of \( A \) generated by a positive element \( z \) of \( A \).

**Lemma 1.** Let \( A \) be a simple \( C^* \)-algebra with FS and \( \alpha \) an outer automorphism of \( A \). Then for any non-zero hereditary \( C^* \)-subalgebra \( B \) of \( A \), it follows that

\[
\inf \{\|\rho\alpha(p)\| : p \text{ is a projection of } B\} = 0.
\]

**Proof.** For any small positive number \( \varepsilon > 0 \), we know from [7, Lemma 1.1] that there is a positive element \( z \) in \( B \) of norm 1 with \( \|z\alpha(z)\| < \varepsilon \). Since \( A \) has FS, we may assume that the spectrum of \( z \) is finite. Let \( p = \chi_{\{1\}}(z) \) so that \( p \) is a projection such that \( z \geq p \). Hence

\[
\|p\alpha(p)\| \leq \|z\alpha(z)\| < \varepsilon.
\]

**Lemma 2.** Let \( \{p_i\}_{i=1}^n \) be finitely many projections in a simple \( C^* \)-algebra \( A \) such that \( \|p_ip_j\| < \frac{1}{2^n}, \ i \neq j \). Then their supremum \( \vee p_i \) (in \( A^{**} \)) is contained in \( A \).

**Proof.** Recall [5, Lemma 2.7] that if \( e \) and \( f \) are projections in a \( C^* \)-algebra \( A \) with \( \|ef\| < 1 \), then \( e \vee f \in A \). So it suffices to show that

\[
\|p_k(p_1 \vee \cdots \vee p_{k-1})\| < 1, \quad 3 \leq k \leq n.
\]

Consider a \( C^* \)-algebra \( A \) as a subalgebra of \( A^{**} = (\pi_u(A)^{\sigma-wk} \subset B(H) \), where \( (\pi_u, \mathcal{H}) \) is the universal representation of \( A \). Then the supremum \( \vee p_i \) of \( \{p_i\} \) is the projection onto the closed subspace \( \{p_1\xi_1 + p_2\xi_2 + \cdots + p_n\xi_n | \xi_i \in \mathcal{H} \}^- \). Let

\[
\xi = p_1\xi_1 + \cdots + p_{k-1}\xi_{k-1}
\]

be a unit vector in \( (p_1 \vee \cdots \vee p_{k-1})\mathcal{H} \). Then for \( \varepsilon = \frac{1}{2^n} > 0 \), we have

\[
\|p_k(p_1 \vee \cdots \vee p_{k-1})\xi\|^2
\]

\[
= \|p_k(p_1\xi_1 + \cdots + p_{k-1}\xi_{k-1})\|^2
\]

\[
= \sum_{i=1}^{k-1} \|p_kp_i\xi_i\|^2 + \sum_{i \neq j} (p_kp_i\xi_i|p_kp_j\xi_i)
\]

\[
\leq \varepsilon^2 \sum_{i=1}^{k-1} \|p_i\xi_i\|^2 + \varepsilon^2 \sum_{i \neq j} \|p_i\xi_i\|\|p_j\xi_j\|.
\]
On the other hand
\[ 1 = \|\xi\|^2 = \left(\sum_{i=1}^{k-1} \|p_i\xi_i\|^2 + \sum_{i\neq j} \langle p_i\xi_i | p_j\xi_j \rangle \right) \]
\[ \geq \sum_{i=1}^{k-1} \|p_i\xi_i\|^2 - \varepsilon \sum_{i\neq j} \|p_i\xi_i\|\|p_j\xi_j\| \]
\[ = \sum_{i=1}^{k-1} \|p_i\xi_i\|^2 - \varepsilon \left( (k-2) \sum_{i=1}^{k-1} \|p_i\xi_i\|^2 - \sum_{i<j} (\|p_i\xi_i\| - \|p_j\xi_j\|)^2 \right) \]
\[ = (1 - (k-2)\varepsilon) \sum_{i=1}^{k-1} \|p_i\xi_i\|^2 + \varepsilon \sum_{i<j} (\|p_i\xi_i\| - \|p_j\xi_j\|)^2 \]
\[ \geq (1 - (k-2)\varepsilon) \sum_{i=1}^{k-1} \|p_i\xi_i\|^2, \]
and hence we have
\[ \sum_{i=1}^{k-1} \|p_i\xi_i\|^2 \leq \frac{1}{1 - (k-2)\varepsilon}, \tag{2} \]
and for all \( i, \ 1 \leq i \leq k - 1, \)
\[ \|p_i\xi_i\| \leq \frac{1}{\sqrt{1 - (k-2)\varepsilon}}. \tag{3} \]

Therefore it follows from (1), (2), and (3) that
\[ \|p_k(p_1 \vee \cdots \vee p_{k-1})\xi\|^2 \leq \varepsilon^2 \frac{1}{1 - (k-2)\varepsilon} + \varepsilon^2 \sum_{i\neq j} \frac{1}{1 - (k-2)\varepsilon} \]
\[ = \frac{\varepsilon^2}{1 - (k-2)\varepsilon} + \frac{(k-1)(k-2)\varepsilon^2}{1 - (k-2)\varepsilon} \]
\[ \leq \frac{\varepsilon^2}{1 - n\varepsilon} + \frac{n^2\varepsilon^2}{1 - n\varepsilon} \]
\[ = \frac{1}{2n^2} + \frac{1}{2} < 1. \]

An action \( \alpha \) of a group \( G \) on a \( C^* \)-algebra \( A \) is said to be outer if each automorphism \( \alpha_g \) is outer for each \( g \neq 1 \), where \( 1 \) denotes the unit of \( G \).

**Theorem 3.** If \( \alpha \) is an outer action by a finite group \( G \) on a simple \( C^* \)-algebra \( A \) with FS, then the crossed product \( A \times_\alpha G \) has SP.

**Proof.** The fixed point algebra \( A^\alpha \) can be identified as a hereditary \( C^* \)-subalgebra of the crossed product \( A \times_\alpha G \) [8] which is simple [7, Theorem 3.1]. If \( B \) is any non-zero hereditary \( C^* \)-subalgebra of \( A \times_\alpha G \), then we can find a unitary \( u \) in the multiplier algebra \( M(A \times_\alpha G) \) of \( A \times_\alpha G \) such that \( uBu^* \cap A^\alpha \neq 0 \) [9, Lemma 3.4] since \( M(A \times_\alpha G) \) is primitive, that is, it does not have two orthogonal non-zero ideals. Therefore it suffices to show that \( A^\alpha \) has SP. For any non-zero positive element \( z \) in \( A^\alpha \) consider the hereditary \( C^* \)-subalgebra
$A_z$ of $A$. Then $A_z$ is invariant under the action $\alpha$. Put $G = \{1, g_1, \ldots, g_n\}$. We can choose a projection $p_1$ in $A_z$ such that $\|p_1\alpha_{g_1}(p_1)\| < \varepsilon$ for sufficiently small $\varepsilon > 0$ by Lemma 1. Since the automorphism $\alpha_{g_2}$ is outer, the hereditary $C^*$-subalgebra $A_{p_1}$ has a projection $p_2$ such that $\|p_2\alpha_{g_2}(p_2)\| < \varepsilon$, so that we have $\|p_2\alpha_{g_1}(p_2)\| < \varepsilon$. By repeating this process, we can take a projection $p$ in $A_z$ satisfying

$$\|(\alpha_s(p)\alpha_t(p))\| < \varepsilon, \quad s \neq t, \ s, t \in G.$$ 

Lemma 2 says that their supremum $P = \vee_{g \in G} \alpha_g(p)$ belongs to $A_z$ since $A_z$ is $\alpha$-invariant. Note that $P$ is the smallest projection $e$ in $A^{**}$ satisfying

$$(*) \quad e(p + \alpha_{g_1}(p) + \cdots + \alpha_{g_n}(p)) = p + \alpha_{g_1}(p) + \cdots + \alpha_{g_n}(p).$$

If a projection $e$ satisfies $(*)$, then clearly so does $\alpha_g(e)$, $g \in G$. Suppose that $\alpha_g(P)$, $g \in G$, has a proper subprojection $e$ for which $(*)$ holds. Then $\alpha_{g^{-1}}(e)$ is a proper subprojection of $P$ satisfying $(*)$, a contradiction. Since $(*)$ also holds for $\alpha_g(P)$, we conclude that $P = \alpha_g(P)$ and $P$ is the desired projection in $(A^a)_z$, a hereditary $C^*$-subalgebra of $A^a$.

**Remark 4.** (1) It is well known that a $C^*$-algebra $A$ has FS if and only if $A \otimes \mathcal{A}$ has FS [4]. As was noted in the proof of Theorem 3, a simple $C^*$-algebra $A$ has SP if and only if $A$ has a non-zero hereditary $C^*$-subalgebra $B$ with SP. Hence it follows that a simple $C^*$-algebra $A$ has SP if and only if $A = \alpha_g(P)$ and $P$ is the desired projection in $(A^a)_z$.

(2) It is not known whether an infinite simple $C^*$-algebra is purely infinite or not. So it would be interesting to investigate the pure infiniteness of the crossed product $A \times_{\alpha} G$ in Theorem 3 when $A$ is a purely infinite simple $C^*$-algebra since $A \times_{\alpha} G$ is an infinite simple $C^*$-algebra. In fact, it suffices to show that each projection $\vee_{g \in G} \alpha_g(p)$ constructed in the proof of Theorem 3 is infinite in $(A^a)_z$.

**REFERENCES**


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