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PROJECTIONS IN SOME SIMPLE C^*-CROSSED PRODUCTS

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ABSTRACT. Let α be an outer action by a finite group G on a simple C^*-algebra A. If each hereditary C^*-subalgebra of A has an approximate identity consisting of projections, then every hereditary C^*-subalgebra of the crossed product $A \times_\alpha G$ has a projection.

1. INTRODUCTION

A C^*-algebra A is said to have **FS** if the set of all selfadjoint elements with finite spectra (as an element of $\tilde{A}$, the unital C^*-algebra obtained by adjoining the unit to A) is norm dense in A_{sa}, equally, if every hereditary C^*-subalgebra of A has an approximate identity consisting of projections [4]. This class of C^*-algebras includes AF-algebras, von Neumann algebras [4], purely infinite simple C^*-algebras [10], etc. A *purely infinite* C^*-algebra is a C^*-algebra such that any of its hereditary C^*-subalgebra is infinite, that is, has an infinite projection. If each hereditary C^*-subalgebra of a C^*-algebra A has a non-zero projection, then we say that A has **SP**. There are many examples of C^*-algebras which do not have **FS** but **SP**. For example, consider a purely infinite C^*-algebra $\mathscr{C} \otimes \mathscr{K}$, where $\mathscr{K}$ is the algebra of compact operators on a separable infinite-dimensional Hilbert space $\mathscr{H}$ and $\mathscr{C}$ is the Calkin algebra $B(\mathscr{H})/\mathscr{K}$. Then its multiplier algebra $M(\mathscr{C} \otimes \mathscr{K})$ is also a purely infinite C^*-algebra which does not have **FS** [11] but has **SP**, since the multiplier algebra of a simple (or primitive, in general) C^*-algebra with **SP** obviously has **SP**.

While there are various examples of projectionless simple C^*-algebras, a large class of simple C^*-algebras are also known to contain projections [2].

In [3] Blackadar and Kumjian construct a simple C^*-algebra which does not have **FS** but **SP**. We do not know whether the two conditions **FS** and **SP** are equivalent or not for infinite simple C^*-algebras. Both conditions mean that a C^*-algebra abounds in its projections so that if A is an infinite-dimensional simple C^*-algebra with **SP** (or **FS**), then A contains no minimal projections.

We show in this short note that the crossed product $A \times_\alpha G$ by an outer action α of a finite group G has **SP** whenever A is a simple C^*-algebra with **FS**. In [6,

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Example 9], Elliott showed that there is an action of $\mathbb{Z}_2$ on a simple C^*-algebra which does not have **FS** but the crossed product does. So by Takesaki-Takai duality, it follows that the property **FS** is not necessarily preserved in crossed products by finite groups.

2. Projections in simple C^*-crossed products

Throughout this paper A_z denotes the hereditary C^*-subalgebra of A generated by a positive element z of A.

Lemma 1. *Let A be a simple C^*-algebra with* **FS** *and α an outer automorphism of A. Then for any non-zero hereditary C^*-subalgebra B of A, it follows that*

$$\inf\{\|p\alpha(p)\| \ : \ p \textit{ is a projection of } B\} = 0.$$

Proof. For any small positive number $\varepsilon > 0$, we know from [7, Lemma 1.1] that there is a positive element z in B of norm 1 with $\|z\alpha(z)\| < \varepsilon$. Since A has **FS**, we may assume that the spectrum of z is finite. Let $p = \chi_{\{1\}}(z)$ so that p is a projection such that $z \geq p$. Hence

$$\|p\alpha(p)\| \leq \|z\alpha(z)\| < \varepsilon.$$

Lemma 2. *Let $\{p_i\}_{i=1}^n$ be finitely many projections in a simple C^*-algebra A such that $\|p_ip_j\| < \frac{1}{2n}$, $i \neq j$. Then their supremum $\vee p_i$ (in A^{**}) is contained in A.*

Proof. Recall [5, Lemma 2.7] that if e and f are projections in a C^*-algebra A with $\|ef\| < 1$, then $e \vee f \in A$. So it suffices to show that

$$\|p_k(p_1 \vee \cdots \vee p_{k-1})\| < 1, \qquad 3 \leq k \leq n.$$

Consider a C^*-algebra A as a subalgebra of A^{**} $(= \overline{\pi_u(A)}^{\sigma-wk} \subset \mathscr{B}(\mathscr{H})$, where $(\pi_u, \mathscr{H})$ is the universal representation of A). Then the supremum $\vee p_i$ of $\{p_i\}$ is the projection onto the closed subspace $\{p_1\xi_1 + p_2\xi_2 + \cdots + p_n\xi_n | \xi_i \in \mathscr{H}\}^-$. Let

$$\xi = p_1\xi_1 + \cdots + p_{k-1}\xi_{k-1}$$

be a unit vector in $(p_1 \vee \cdots \vee p_{k-1})\mathscr{H}$. Then for $\varepsilon = \frac{1}{2n} > 0$, we have

$$\begin{aligned} &\|p_k(p_1 \vee \cdots \vee p_{k-1})\xi\|^2 \\ &\quad = \|p_k(p_1\xi_1 + \cdots + p_{k-1}\xi_{k-1})\|^2 \\ &\quad = \sum_{i=1}^{k-1} \|p_kp_i\xi_i\|^2 + \sum_{i\neq j} \langle p_kp_i\xi_i | p_kp_j\xi_j\rangle \qquad (1) \\ &\quad \leq \varepsilon^2 \sum_{i=1}^{k-1} \|p_i\xi_i\|^2 + \varepsilon^2 \sum_{i\neq j} \|p_i\xi_i\|\|p_j\xi_j\|. \end{aligned}$$

On the other hand

$$
\begin{aligned}
1 = \|\xi\|^2 = \|\sum_{i=1}^{k-1} p_i\xi_i\|^2 &= \sum_{i=1}^{k-1}\|p_i\xi_i\|^2 + \sum_{i\neq j}\langle p_i\xi_i|p_j\xi_j\rangle \\
&\geq \sum_{i=1}^{k-1}\|p_i\xi_i\|^2 - \varepsilon\sum_{i\neq j}\|p_i\xi_i\|\|p_j\xi_j\| \\
&= \sum_{i=1}^{k-1}\|p_i\xi_i\|^2 - \varepsilon\left\{(k-2)\sum_{i=1}^{k-1}\|p_i\xi_i\|^2 - \sum_{i<j}(\|p_i\xi_i\| - \|p_j\xi_j\|)^2\right\} \\
&= (1-(k-2)\varepsilon)\sum_{i=1}^{k-1}\|p_i\xi_i\|^2 + \varepsilon\sum_{i<j}(\|p_i\xi_i\| - \|p_j\xi_j\|)^2 \\
&\geq (1-(k-2)\varepsilon)\sum_{i=1}^{k-1}\|p_i\xi_i\|^2,
\end{aligned}
$$

and hence we have

$$\sum_{i=1}^{k-1}\|p_i\xi_i\|^2 \leq \frac{1}{1-(k-2)\varepsilon}, \tag{2}$$

and for all i, $1 \leq i \leq k-1$,

$$\|p_i\xi_i\| \leq \frac{1}{\sqrt{1-(k-2)\varepsilon}}. \tag{3}$$

Therefore it follows from (1), (2), and (3) that

$$
\begin{aligned}
\|p_k(p_1 \vee \cdots \vee p_{k-1})\xi\|^2 &\leq \varepsilon^2\frac{1}{1-(k-2)\varepsilon} + \varepsilon^2\sum_{i\neq j}\frac{1}{1-(k-2)\varepsilon} \\
&= \frac{\varepsilon^2}{1-(k-2)\varepsilon} + \frac{(k-1)(k-2)\varepsilon^2}{1-(k-2)\varepsilon} \\
&\leq \frac{\varepsilon^2}{1-n\varepsilon} + \frac{n^2\varepsilon^2}{1-n\varepsilon} \\
&= \frac{1}{2n^2} + \frac{1}{2} < 1.
\end{aligned}
$$

An action α of a group G on a C^*-algebra A is said to be *outer* if each automorphism α_g is outer for each $g \neq 1$, where 1 denotes the unit of G.

Theorem 3. *If α is an outer action by a finite group G on a simple C^*-algebra A with* **FS**, *then the crossed product $A \times_\alpha G$ has* **SP**.

Proof. The fixed point algebra A^α can be identified as a hereditary C^*-subalgebra of the crossed product $A \times_\alpha G$ [8] which is simple [7, Theorem 3.1]. If B is any non-zero hereditary C^*-subalgebra of $A \times_\alpha G$, then we can find a unitary u in the multiplier algebra $M(A \times_\alpha G)$ of $A \times_\alpha G$ such that $uBu^* \cap A^\alpha \neq 0$ [9, Lemma 3.4] since $M(A \times_\alpha G)$ is primitive, that is, it does not have two orthogonal non-zero ideals. Therefore it suffices to show that A^α has **SP**. For any non-zero positive element z in A^α consider the hereditary C^*-subalgebra

A_z of A. Then A_z is invariant under the action α. Put $G = \{1, g_1, \dots, g_n\}$. We can choose a projection p_1 in A_z such that $\|p_1\alpha_{g_1}(p_1)\| < \varepsilon$ for sufficiently small $\varepsilon > 0$ by Lemma 1. Since the automorphism α_{g_2} is outer, the hereditary C^*-subalgebra A_{p_1} has a projection p_2 such that $\|p_2\alpha_{g_2}(p_2)\| < \varepsilon$, so that we have $\|p_2\alpha_{g_1}(p_2)\| < \varepsilon$. By repeating this process, we can take a projection p in A_z satisfying

$$\|\alpha_s(p)\alpha_t(p)\| < \varepsilon, \qquad s \neq t, \; s, t \in G.$$

Lemma 2 says that their supremum $P = \bigvee_{g\in G} \alpha_g(p)$ belongs to A_z since A_z is α-invariant. Note that P is the smallest projection e in A^{**} satisfying

$$(*) \qquad e(p + \alpha_{g_1}(p) + \cdots + \alpha_{g_n}(p)) = p + \alpha_{g_1}(p) + \cdots + \alpha_{g_n}(p).$$

If a projection e satisfies $(*)$, then clearly so does $\alpha_g(e)$, $g \in G$. Suppose that $\alpha_g(P)$, $g \in G$, has a proper subprojection e for which $(*)$ holds. Then $\alpha_{g^{-1}}(e)$ is a proper subprojection of P satisfying $(*)$, a contradiction. Since $(*)$ also holds for $\alpha_g(P)$, we conclude that $P = \alpha_g(P)$ and P is the desired projection in $(A^\alpha)_z$, a hereditary C^*-subalgebra of A^α.

Remark 4. (1) It is well known that a C^*-algebra A has **FS** if and only if $A\otimes\mathscr{K}$ has **FS** [4]. As was noted in the proof of Theorem 3, a simple C^*-algebra A has **SP** if and only if A has a non-zero hereditary C^*-subalgebra B with **SP**. Hence it follows that a simple C^*-algebra A has **SP** if and only if $A \otimes \mathscr{K}$ has **SP**.

(2) It is not known whether an infinite simple C^*-algebra is purely infinite or not. So it would be interesting to investigate the pure infiniteness of the crossed product $A \times_\alpha G$ in Theorem 3 when A is a purely infinite simple C^*-algebra since $A \times_\alpha G$ is an infinite simple C^*-algebra. In fact, it suffices to show that each projection $\bigvee_{g\in G} \alpha_g(p)$ constructed in the proof of Theorem 3 is infinite in $(A^\alpha)_z$.

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