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ON THE FRIEDRICHS OPERATOR

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ABSTRACT. Let Ω be a simply connected domain in $\mathbb{C}^1$ with the area measure dA. Let $\overline{P}_\Omega$ be the orthogonal projection from $L^2(\Omega, dA)$ onto the closed subspace of antiholomorphic functions in $L^2(\Omega, dA)$. The Friedrichs operator $\overline{T}_\Omega$ associated to Ω is the operator from the Bergman space $L_a^2(\Omega)$ into $L^2(\Omega, dA)$ defined by $\overline{T}_\Omega f = \overline{P}_\Omega f$. In this note, some smoothness conditions on the boundary of Ω are given such that the Friedrichs operator $\overline{T}_\Omega$ belongs to the Schatten classes S_p.

1. INTRODUCTION

Let Ω be a connected open set in the complex plane $\mathbb{C}^1$ and let dA denote the area measure. Let $L^2(\Omega) = L^2(\Omega, dA)$ be the usual Lebesgue space, $L_a^2(\Omega)$ the Bergman space consisting of holomorphic functions in $L^2(\Omega)$ and $\overline{L_a^2(\Omega)} = \{\bar{f} : f \in L_a^2(\Omega)\}$. Let P_Ω be the orthogonal projection from $L^2(\Omega)$ onto $L_a^2(\Omega)$ and $\overline{P}_\Omega$ the orthogonal projecton from $L^2(\Omega)$ onto $\overline{L_a^2(\Omega)}$. Let $K_\Omega(z, w)$ denote the Bergman kernel of Ω. Then

$$\begin{aligned} P_\Omega f(z) &= \int_\Omega K_\Omega(z, w) f(w)\, dA(w), \\ \overline{P}_\Omega f(z) &= \int_\Omega \overline{K_\Omega(z, w)} f(w)\, dA(w) \\ &= \int_\Omega K_\Omega(w, z) f(w)\, dA(w). \end{aligned}$$

According to [Sh2], the Friedrichs operator associated to Ω is defined by

$$T_\Omega f = P_\Omega \bar{f} \qquad \text{for} \quad f \in L_a^2(\Omega).$$

Note that T_Ω is bounded and real-linear, but not complex-linear since

$$T_\Omega(\lambda f) = \bar{\lambda} T_\Omega f, \qquad \lambda \in \mathbb{C}.$$

Observe that $\overline{T_\Omega f} = \overline{P_\Omega \bar{f}} = \overline{P}_\Omega f$. We introduce the operator $\overline{T}_\Omega$ which is

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defined by

$$\overline{T}_\Omega f = \overline{T_\Omega f} = \overline{P}_\Omega f \qquad \text{for} \quad f \in L^2_a(\Omega).$$

Then $\overline{T}_\Omega$ is a complex-linear operator. It is obvious that T^2_Ω is also a complex-linear operator.

If $\Omega = \mathbb{D}$, the unit disk, then it is easy to check that $T_\mathbb{D}$ (and $\overline{T}_\mathbb{D}$ too) is a rank-one operator. But for general Ω, T_Ω (or $\overline{T}_\Omega$) may even not be compact. In 1937, Friedrichs proved the following compactness theorem.

Theorem (Friedrichs, see [Sh2]). *If Ω is a bounded connected domain in $\mathbb{C}^1$ with $C^{1,\alpha}$ boundary, then T_Ω (or equivalently $\overline{T}_\Omega$) is compact.*

In [Sh1], Shapiro gave a characterization of the finite-rank Friedrichs operator T_Ω (or equivalently $\overline{T}_\Omega$) by using the notion of quadrature domain. In [F] and [N], Friedrichs and Norman studied the essential spectrum of T^2_Ω in cases where T^2_Ω is not compact. As one can see from the above, the spectral properties of the Friedrichs operator are closely related to the geometry of Ω.

The problem that we are interested in here is the following.

Problem. *When are $\overline{T}_\Omega$ and T^2_Ω in the Schatten classes S_p?*

In this note we will concentrate on simply connected domains in $\mathbb{C}^1$. To state our main results, we need some definitions and notation.

The Schatten class S_p ($1 \le p < \infty$) consists of all the compact operators T from a Hilbert space H_1 to another Hilbert space H_2 for which the singular numbers $s_n(T)$ form a sequence belonging to l^p. The singular numbers of the operator T are the eigenvalues of $(T^*T)^{1/2}$.

For $s \in \mathbb{Z}_+$, $\alpha > 0$ and $1 \le p < \infty$, the weighted Sobolev space $W^{s,\alpha}_p(\mathbb{D})$ on the unit disk is defined to be the completion of $C^\infty(\overline{\mathbb{D}})$ with respect to the norm

$$\|f\|_{W^{s,\alpha}_p(\mathbb{D})} = \left(\sum_{\substack{\xi+\eta \leqslant s \\ \xi,\eta \in \mathbb{Z}_+}} \left\| \left(\frac{\partial}{\partial z}\right)^\xi \left(\frac{\partial}{\partial \bar z}\right)^\eta f \right\|^p_{L^p(\mathbb{D},\,(1-|z|^2)^{\alpha-1} dA(z))} \right)^{1/p}.$$

Note that when $\alpha = 1$, $W^{s,1}_p(\mathbb{D}) = W^s_p(\mathbb{D})$, which is the usual Sobolev space.

A Jordan curve Γ is of class $C^{n,\alpha}$ ($n = 1, 2, \cdots, 0 < \alpha < 1$) if it has a parametrization $\Gamma : w(\tau) : 0 \le \tau \le 2\pi$ that is n times continuously differentiable and satisfies $w'(\tau) \ne 0$ and $|w^{(n)}(\tau_1) - w^{(n)}(\tau_2)| \le C|\tau_1 - \tau_2|^\alpha$.

Main Theorem. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. Then:*

(1) *For $1 < p < \infty$, if $\varphi'/\overline{\varphi'} \in W^{1,p-1}_p(\mathbb{D})$, then $\overline{T}_\Omega \in S_p$.*
(2) *For $p = 1$, if $\varphi'/\overline{\varphi'} \in W^{2,\gamma}_1(\mathbb{D})$ for some $0 < \gamma < 1$, then $\overline{T}_\Omega \in S_1$.*

From the Main Theorem we have

Corollary A. *If Ω is a bounded simply connected domain in $\mathbb{C}^1$ with $C^{2,\alpha}$ ($0 < \alpha < 1$) boundary, then $\overline{T}_\Omega$ belongs to all S_p for $1 < p < \infty$.*

Corollary B. *If Ω is a bounded simply connected domain in $\mathbb{C}^1$ with $C^{3,\alpha}$ ($0 < \alpha < 1$) boundary, then $\overline{T}_\Omega \in S_1$.*

For T^2_Ω we have similar results.

2. Compactness

In this section we will give some other conditions for the compactness of the Friedrichs operator. For the bounded simply connected domain case, Friedrichs' compactness theorem [F] will be a corollary of our results.

As we will see in the following lemma, the Friedrichs operator is closely related to the small Hankel operator h_b on the unit disk $\mathbb{D}$. Given $b \in L^2(\mathbb{D})$, the small Hankel operator h_b (with symbol b) is defined by

$$h_b f = \overline{P}_{\mathbb{D}}(bf) \qquad \text{for} \quad f \in L^2_a(\mathbb{D}).$$

Lemma 2.1. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. Then*

$$\overline{T}_\Omega = V_{\varphi^{-1}} h_{\overline{\varphi'}/\varphi'} U_\varphi = V_{\varphi^{-1}} h_{\overline{P_{\mathbb{D}}(\varphi'/\overline{\varphi'})}} U_\varphi,$$

where $U_\varphi : L^2_a(\Omega) \to L^2_a(\mathbb{D})$, $U_\varphi f = f \circ \varphi \cdot \varphi'$ and $V_{\varphi^{-1}} : \overline{L^2_a(\mathbb{D})} \to \overline{L^2_a(\Omega)}$, $V_{\varphi^{-1}} g = g \circ \varphi^{-1} \cdot \overline{(\varphi^{-1})'}$.

Proof. From p. 33 of [B] we have

$$K_\Omega(w, z) = K_{\mathbb{D}}(\varphi^{-1}(w), \varphi^{-1}(z))(\varphi^{-1})'(w)\overline{(\varphi^{-1})'(z)}.$$

Then for any $f \in L^2_a(\Omega)$,

$$\begin{aligned}
\overline{T}_\Omega f(z) = \overline{P}_\Omega f(z) &= \int_\Omega K_\Omega(w, z) f(w)\, dA(w) \\
&= \int_\Omega K_{\mathbb{D}}(\varphi^{-1}(w), \varphi^{-1}(z))(\varphi^{-1})'(w)\overline{(\varphi^{-1})'(z)} f(w)\, dA(w) \\
&= \int_{\mathbb{D}} K_{\mathbb{D}}(w, \varphi^{-1}(z)) \frac{1}{\varphi'(w)} \overline{(\varphi^{-1})'(z)} f \circ \varphi(w) |\varphi'(w)|^2\, dA(w) \\
&\qquad \text{(by change of variable)} \\
&= \overline{(\varphi^{-1})'(z)} \int_{\mathbb{D}} K_{\mathbb{D}}(w, \varphi^{-1}(z)) \frac{\overline{\varphi'(w)}}{\varphi'(w)} f \circ \varphi(w) \varphi'(w)\, dA(w) \\
&= (V_{\varphi^{-1}} h_{\overline{\varphi'}/\varphi'} U_\varphi f)(z).
\end{aligned}$$

Thus

$$\overline{T}_\Omega = V_{\varphi^{-1}} h_{\overline{\varphi'}/\varphi'} U_\varphi.$$

The second equality follows from the following property of the small Hankel operator:

$$h_{\bar b} = h_{\overline{P_{\mathbb{D}} b}} \qquad \text{for} \quad b \in L^\infty(\mathbb{D}). \qquad \square$$

By using Lemma 2.1, we can get the following compactness result.

Theorem 2.2. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. Then T_Ω is compact if and only if $P_{\mathbb{D}}(\varphi'/\overline{\varphi'}) \in \mathscr{B}_0(\mathbb{D})$, where $\mathscr{B}_0(\mathbb{D})$ is the little Bloch space which consists of holomorphic functions f on $\mathbb{D}$ such that $(1-|z|^2)f'(z) \to 0$ as $|z| \to 1^-$.*

Proof. Since T_Ω is compact if and only if $\overline{T}_\Omega$ is compact, we only need to consider $\overline{T}_\Omega$.

Since $U_\varphi : L_a^2(\Omega) \to L_a^2(\mathbb{D})$ and $V_{\varphi^{-1}} : \overline{L_a^2(\mathbb{D})} \to \overline{L_a^2(\Omega)}$ are unitary operators, by Lemma 2.1 we have that $\overline{T}_\Omega$ is compact if and only if the small Hankel operator $h_{\overline{P_{\mathbb{D}}(\varphi'/\overline{\varphi'})}}$ is compact. But by Corollary 2 of Theorem 7.6.6 in [Zh], we know that $h_{\overline{P_{\mathbb{D}}(\varphi'/\overline{\varphi'})}}$ is compact if and only if $P_{\mathbb{D}}(\varphi'/\overline{\varphi'}) \in \mathscr{B}_0(\mathbb{D})$. This completes the proof. □

From Theorem 2.2 we have

Theorem 2.3. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. If* $\arg(\varphi') \in C(\overline{\mathbb{D}})$, *then T_Ω is compact.*

Proof. If $\arg(\varphi') \in C(\overline{\mathbb{D}})$, we then have $\varphi'/\overline{\varphi'} = e^{2i\arg(\varphi')} \in C(\overline{\mathbb{D}})$. By Theorem 5.2.5 of [Zh], $P_{\mathbb{D}}$ maps $C(\overline{\mathbb{D}})$ onto $\mathscr{B}_0(\mathbb{D})$. Thus $P_{\mathbb{D}}(\varphi'/\overline{\varphi'}) \in \mathscr{B}_0(\mathbb{D})$. Then by Theorem 2.2, T_Ω is compact. □

As a corollary of Theorem 2.3, we can get the following Friedrichs' compactness theorem [F].

Corollary 2.4 ([F]). *If Ω is a bounded simply connected domain in $\mathbb{C}^1$ with C^1 boundary, then T_Ω is compact.*

The proof of Corollary 2.4 follows immediately from Theorem 2.3 and the following lemma.

Lemma 2.5. *Let φ map $\mathbb{D}$ conformally onto the inner domain of the Jordan curve Γ. Then Γ is of class C^1 if and only if* $\arg(\varphi')$ *has a continuous extension to $\overline{\mathbb{D}}$.*

Proof. See Theorem 3.2 of [Po]. □

Given $b \in L^\infty(\mathbb{D})$, the big Hankel operator H_b (with symbol b) is defined by $H_b f = (I - P_{\mathbb{D}})(bf)$ for $f \in L_a^2(\mathbb{D})$. Since the small Hankel operator h_b is compact whenever the big Hankel operator H_b is, by using the compactness result of the big Hankel operator H_b, we can also obtain the following sufficient condition for the compactness of the Friedrichs operator T_Ω.

Theorem 2.6. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. If $\log\varphi' \in \mathscr{B}_0(\mathbb{D})$, then T_Ω is compact.*

Proof. Equivalently, we only need to prove that $\overline{T}_\Omega$ is compact. By Lemma 2.1, we know that $\overline{T}_\Omega$ is compact if and only if the small Hankel operator $h_{\overline{\varphi'}/\varphi'}$ is compact. Now we consider the big Hankel operator $H_{\overline{\varphi'}/\varphi'}$. Since the symbol function $\overline{\varphi'}/\varphi'$ is smooth, by Theorem 2 of [Lu] we have that $H_{\overline{\varphi'}/\varphi'}$ is compact if the following condition is satisfied:

$$(2.1) \qquad (1-|z|^2)\frac{\partial}{\partial \bar z}\left(\frac{\overline{\varphi'}}{\varphi'}\right)(z) \to 0 \quad \text{as} \quad |z| \to 1.$$

But

$$\left|\frac{\partial}{\partial \bar z}\frac{\overline{\varphi'}}{\varphi'}\right| = \left|\frac{\overline{\varphi''}\varphi'}{(\varphi')^2}\right| = \left|\frac{\varphi''}{\varphi'}\right| = |(\log\varphi')'|.$$

Therefore condition (2.1) is equivalent to $\log\varphi' \in \mathscr{B}_0(\mathbb{D})$. Thus, if $\log\varphi' \in \mathscr{B}_0(\mathbb{D})$, then the big Hankel operator $H_{\overline{\varphi'}/\varphi'}$ is compact, and this implies that the small Hankel operator $h_{\overline{\varphi'}/\varphi'}$ is compact. Hence $\overline{T}_\Omega$ is compact. This completes the proof. □

3. SCHATTEN CLASS FRIEDRICHS OPERATORS

First, we need to collect a couple of definitions.

For $1 < p < \infty$, the holomorphic Besov space $B_p(\mathbb{D})$ on the unit disk is defined to be the space of holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{B_p(\mathbb{D})} = \left(\int_{\mathbb{D}} (1-|z|^2)^{p-2}|f'(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

For $p = 1$, $B_1(\mathbb{D})$ is the space of holomorphic functions f on $\mathbb{D}$ which can be written as

$$f(z) = \sum_{n=1}^{+\infty} a_n \frac{\lambda_n - z}{1 - \bar{\lambda}_n z}$$

for some sequence $\{a_n\}$ in l^1 and $\{\lambda_n\}$ in $\mathbb{D}$.

The weighted holomorphic Sobolev space $\mathscr{W}_p^{s,\alpha}(\mathbb{D})$ on the unit disk is the subspace of $W_p^{s,\alpha}(\mathbb{D})$ (defined in Section 1) consisting of holomorphic functions on $\mathbb{D}$.

Now we can consider the membership of $\overline{T}_\Omega$ in the Schatten classes S_p.

Theorem 3.1. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. Then for $1 \le p < \infty$, $\overline{T}_\Omega \in S_p$ if and only if $P_{\mathbb{D}}(\varphi'/\overline{\varphi'}) \in B_p(\mathbb{D})$.*

Proof. Since $U_\varphi : L_a^2(\Omega) \to L_a^2(\mathbb{D})$ and $V_{\varphi^{-1}} : \overline{L_a^2(\mathbb{D})} \to \overline{L_a^2(\Omega)}$ are unitary operators, by Lemma 2.1 we have that $\overline{T}_\Omega \in S_p$ if and only if $h_{\overline{P_{\mathbb{D}}(\varphi'/\overline{\varphi'})}} \in S_p$. But by Corollary 2 of Theorem 7.6.7 in [Zh], we know that $h_{\overline{P_{\mathbb{D}}(\varphi'/\overline{\varphi'})}} \in S_p$ if and only if $P_{\mathbb{D}}(\varphi'/\overline{\varphi'}) \in B_p(\mathbb{D})$. This completes the proof. □

In order to get our main theorem, we need the following lemma.

Lemma 3.2. *Let $s \in \mathbb{Z}_+$ and $1 \le p < \infty$. Then for any $0 < \gamma < p$, $P_{\mathbb{D}}$ projects $W_p^{s,\gamma}(\mathbb{D})$ continuously onto its holomorphic subspace $\mathscr{W}_p^{s,\gamma}(\mathbb{D})$.*

Proof. See [BB]. This lemma is a special case of Corollary 6.3(ii) in [BB]. □

Now we are ready for our main theorem.

Main Theorem. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. Then:*

(1) *For $1 < p < \infty$, if $\varphi'/\overline{\varphi'} \in W_p^{1,p-1}(\mathbb{D})$, then $\overline{T}_\Omega \in S_p$.*

(2) *For $p = 1$, if $\varphi'/\overline{\varphi'} \in W_1^{2,\gamma}(\mathbb{D})$ for some $0 < \gamma < 1$, then $\overline{T}_\Omega \in S_1$.*

Proof. First we prove (1). For $1 < p < \infty$, using Lemma 3.2, we have

$$\|P_{\mathbb{D}} f\|_{\mathscr{W}_p^{1,p-1}(\mathbb{D})} \le C\|f\|_{W_p^{1,p-1}(\mathbb{D})} \qquad \text{for} \quad f \in W_p^{1,p-1}(\mathbb{D}).$$

Then if $\varphi'/\overline{\varphi'} \in W_p^{1,p-1}(\mathbb{D})$, we have

$$\begin{aligned}\|P_{\mathbb{D}}(\varphi'/\overline{\varphi'})\|_{B_p(\mathbb{D})} &= \left(\int_{\mathbb{D}} \left|\frac{\partial}{\partial z}(P_{\mathbb{D}}(\varphi'/\overline{\varphi'}))\right|^p (1-|z|^2)^{p-2}\, dA\right)^{1/p} \\ &\le \|P_{\mathbb{D}}(\varphi'/\overline{\varphi'})\|_{\mathscr{W}_p^{1,p-1}(\mathbb{D})} \\ &\le C\|\varphi'/\overline{\varphi'}\|_{W_p^{1,p-1}(\mathbb{D})} < \infty.\end{aligned}$$

Thus $P_{\mathbb{D}}(\varphi'/\overline{\varphi'}) \in B_p(\mathbb{D})$. It then follows from Theorem 3.1 that $\overline{T}_\Omega \in S_p$.

Now we prove (2). For $p = 1$, using Lemma 3.2, we have for any $0 < \gamma < 1$,

$$\|P_{\mathbb{D}} f\|_{\mathscr{W}_1^{2,\gamma}(\mathbb{D})} \le C\|f\|_{W_1^{2,\gamma}(\mathbb{D})} \qquad \text{for} \quad f \in W_1^{2,\gamma}(\mathbb{D}).$$

On the other hand, it is well known that $g \in B_1(\mathbb{D})$ if and only if

$$\int_{\mathbb{D}} |g''(z)|\, dA(z) < \infty.$$

Then if $\varphi'/\overline{\varphi'} \in W_1^{2,\gamma}(\mathbb{D})$ for some $0 < \gamma < 1$, we have

$$\begin{aligned}\int_{\mathbb{D}} \left|\frac{\partial^2}{\partial z^2}(P_{\mathbb{D}}(\varphi'/\overline{\varphi'}))\right| dA(z) &\le \int_{\mathbb{D}} \left|\frac{\partial^2}{\partial z^2}(P_{\mathbb{D}}(\varphi'/\overline{\varphi'}))\right| (1-|z|^2)^{\gamma-1}\, dA(z) \\ &\le \|P_{\mathbb{D}}(\varphi'/\overline{\varphi'})\|_{\mathscr{W}_1^{2,\gamma}(\mathbb{D})} \\ &\le C\|\varphi'/\overline{\varphi'}\|_{W_1^{2,\gamma}(\mathbb{D})} < \infty.\end{aligned}$$

Thus $P_{\mathbb{D}}(\varphi'/\overline{\varphi'}) \in B_1(\mathbb{D})$. It then follows from Theorem 3.1 that $\overline{T}_\Omega \in S_1$. This completes the proof of (2) and of the theorem. □

From the Main Theorem we have several corollaries.

Corollary 3.3. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. Then for $1 < p < \infty$, if $\log \varphi' \in B_p(\mathbb{D})$, then $\overline{T}_\Omega \in S_p$.*

Proof. If $\log\varphi' \in B_p(\mathbb{D})$ with $1 < p < \infty$, then

$$\begin{aligned}\|\varphi'/\overline{\varphi'}\|^p_{W_p^{1,p-1}(\mathbb{D})} &= \int_{\mathbb{D}} (1-|z|^2)^{p-2} |\varphi'/\overline{\varphi'}|^p\, dA(z) \\ &\quad + \int_{\mathbb{D}} (1-|z|^2)^{p-2} \left|\frac{\partial}{\partial z}(\varphi'/\overline{\varphi'})\right|^p dA(z) \\ &\quad + \int_{\mathbb{D}} (1-|z|^2)^{p-2} \left|\frac{\partial}{\partial \bar z}(\varphi'/\overline{\varphi'})\right|^p dA(z) \\ &= C + \int_{\mathbb{D}} (1-|z|^2)^{p-2} \left|\frac{\varphi''\overline{\varphi'}}{\overline{\varphi'}^2}\right|^p dA(z) \\ &\quad + \int_{\mathbb{D}} (1-|z|^2)^{p-2} \left|\frac{-\overline{\varphi''}\varphi'}{\overline{\varphi'}^2}\right|^p dA(z) \\ &= C + 2\int_{\mathbb{D}} (1-|z|^2)^{p-2} \left|\frac{\varphi''}{\varphi'}\right|^p dA(z) \\ &= C + 2\|\log\varphi'\|^p_{B_p(\mathbb{D})} < \infty.\end{aligned}$$

Thus $\varphi'/\overline{\varphi'} \in W_p^{1,p-1}(\mathbb{D})$. It then follows from (1) of the Main Theorem that $\overline{T}_\Omega \in S_p$. □

Corollary A. *If Ω is a bounded simply connected domain in $\mathbb{C}^1$ with $C^{2,\alpha}$ $(0 < \alpha < 1)$ boundary, then $\overline{T}_\Omega$ belongs to all S_p for $1 < p < \infty$.*

To prove Corollary A, we need the following lemma.

Lemma 3.4. *Let φ map $\mathbb{D}$ conformally onto the inner domain of the Jordan curve Γ of class $C^{n,\alpha}$ where $n = 1, 2, \cdots$ and $0 < \alpha < 1$. Then $\varphi^{(n)}$ has a continuous extension to $\overline{\mathbb{D}}$ and $\varphi'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$.*

Proof. See Theorems 3.5 and 3.6 of [Po]. □

Proof of Corollary A. Let φ be the conformal mapping of $\mathbb{D}$ onto Ω. By Lemma 3.4, we have that $\varphi \in C^2(\overline{\mathbb{D}})$ and $\varphi'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$. Hence $\varphi'/\overline{\varphi'} \in C^1(\overline{\mathbb{D}})$. It is easy to check that $C^1(\overline{\mathbb{D}}) \subset W_p^{1,p-1}(\mathbb{D})$ $(1 < p < \infty)$. It then follows from (1) of the Main Theorem that $\overline{T}_\Omega \in S_p$ for all $1 < p < \infty$. □

Corollary B. *If Ω is a bounded simply connected domain in $\mathbb{C}^1$ with $C^{3,\alpha}$ $(0 < \alpha < 1)$ boundary, then $\overline{T}_\Omega \in S_1$.*

Proof. Let φ be the conformal mapping of $\mathbb{D}$ onto Ω. By Lemma 3.4, we have that $\varphi \in C^3(\overline{\mathbb{D}})$ and $\varphi'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$. Hence $\varphi'/\overline{\varphi'} \in C^2(\overline{\mathbb{D}})$. It is easy to see that $C^2(\overline{\mathbb{D}}) \subset W_1^{2,\gamma}(\mathbb{D})$ for any $0 < \gamma < 1$. It then follows from (2) of the Main Theorem that $\overline{T}_\Omega \in S_1$. □

4. About T_Ω^2

As we pointed out in the introduction, T_Ω is real-linear but not complex-linear. Sometimes (as in [F] and [N]) it is convenient to study T_Ω^2 instead of T_Ω. T_Ω^2 is a complex-linear positive (hence in particular selfadjoint) operator on $L_a^2(\Omega)$. When T_Ω is compact, T_Ω^2 has a sequence of eigenvalues $1 = \lambda_0 > \lambda_1 \geq \lambda_2 \cdots$ decreasing to zero, and a corresponding sequence of eigenfunctions $\{\varphi_n(z)\}_{n\geq 0}$ which form an orthonormal basis for $L_a^2(\Omega)$ and satisfy the remarkable "double orthogonality" relations

$$\int_\Omega \varphi_m \overline{\varphi}_n \, dA = 0 = \int_\Omega \varphi_m \varphi_n \, dA \qquad (m \neq n).$$

See [Sh2] for a more detailed discussion about T_Ω^2.

Observe that for $f \in L_a^2(\Omega)$, $T_\Omega^2 f = P_\Omega(\overline{T_\Omega f}) = P_\Omega(\overline{T}_\Omega f)$. Hence $T_\Omega^2 = P_\Omega \overline{T}_\Omega$. Thus $T_\Omega^2 \in S_p$ whenever $\overline{T}_\Omega \in S_p$. Then from our main theorem we have

Theorem 4.1. *Let Ω be a proper, simply connected domain in $\mathbb{C}^1$ and let φ be the conformal mapping of $\mathbb{D}$ onto Ω. Then*:

(1) *For $1 < p < \infty$, if $\varphi'/\overline{\varphi'} \in W_p^{1,p-1}(\mathbb{D})$, then $T_\Omega^2 \in S_p$.*

(2) *For $p = 1$, if $\varphi'/\overline{\varphi'} \in W_1^{2,\gamma}(\mathbb{D})$ for some $0 < \gamma < 1$, then $T_\Omega^2 \in S_1$.*

Similarly, Corollaries 3.3, A and B are also valid for T_Ω^2.

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