ON THE FRIEDRICH'S OPERATOR

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ABSTRACT. Let $\Omega$ be a simply connected domain in $\mathbb{C}^1$ with the area measure $dA$. Let $\overline{P}_\Omega$ be the orthogonal projection from $L^2(\Omega, dA)$ onto the closed subspace of antiholomorphic functions in $L^2(\Omega, dA)$. The Friedrichs operator $T_\Omega$ associated to $\Omega$ is the operator from the Bergman space $L^2_b(\Omega)$ into $L^2(\Omega, dA)$ defined by $T_\Omega f = \overline{P}_\Omega f$. In this note, some smoothness conditions on the boundary of $\Omega$ are given such that the Friedrichs operator $T_\Omega$ belongs to the Schatten classes $S_p$.

1. Introduction

Let $\Omega$ be a connected open set in the complex plane $\mathbb{C}^1$ and let $dA$ denote the area measure. Let $L^2(\Omega) = L^2(\Omega, dA)$ be the usual Lebesgue space, $L^2_b(\Omega)$ the Bergman space consisting of holomorphic functions in $L^2(\Omega)$ and $\overline{L}^2_b(\Omega) = \{ \overline{f} : f \in L^2_b(\Omega) \}$. Let $P_\Omega$ be the orthogonal projection from $L^2(\Omega)$ onto $L^2_b(\Omega)$ and $\overline{P}_\Omega$ the orthogonal projection from $L^2(\Omega)$ onto $\overline{L}^2_b(\Omega)$. Let $K_\Omega(z, w)$ denote the Bergman kernel of $\Omega$. Then

$$P_\Omega f(z) = \int_{\Omega} K_\Omega(z, w)f(w) dA(w),$$

$$\overline{P}_\Omega f(z) = \int_{\Omega} K_\Omega(z, w)f(w) dA(w)$$

$$= \int_{\Omega} K_\Omega(w, z)f(w) dA(w).$$

According to [Sh2], the Friedrichs operator associated to $\Omega$ is defined by

$$T_\Omega f = P_\Omega f \quad \text{for} \quad f \in L^2_b(\Omega).$$

Note that $T_\Omega$ is bounded and real-linear, but not complex-linear since

$$T_\Omega(\lambda f) = \overline{\lambda} T_\Omega f, \quad \lambda \in \mathbb{C}.$$ 

Observe that $\overline{T}_\Omega f = \overline{P}_\Omega f = \overline{P}_\Omega f$. We introduce the operator $\overline{T}_\Omega$ which is
defined by
\[ T_\Omega f = \overline{T_\Omega f} = \overline{P_\Omega f} \quad \text{for} \quad f \in L^2_\alpha(\Omega). \]

Then \( T_\Omega \) is a complex-linear operator. It is obvious that \( T_\Omega^2 \) is also a complex-linear operator.

If \( \Omega = \mathbb{D} \), the unit disk, then it is easy to check that \( T_\mathbb{D} \) (and \( T_\mathbb{D} \) too) is a rank-one operator. But for general \( \Omega \), \( T_\Omega \) (or \( T_\Omega \)) may even not be compact. In 1937, Friedrichs proved the following compactness theorem.

Theorem (Friedrichs, see [Sh2]). If \( \Omega \) is a bounded connected domain in \( \mathbb{C}^1 \) with \( C^1, \alpha \) boundary, then \( T_\Omega \) (or equivalently \( T_\Omega \)) is compact.

In [Sh1], Shapiro gave a characterization of the finite-rank Friedrichs operator \( T_\Omega \) (or equivalently \( T_\Omega \)) by using the notion of quadrature domain. In [F] and [N], Friedrichs and Norman studied the essential spectrum of \( T_\Omega^2 \) in cases where \( T_\Omega \) is not compact. As one can see from the above, the spectral properties of the Friedrichs operator are closely related to the geometry of \( \Omega \).

The problem that we are interested in here is the following.

Problem. When are \( T_\Omega \) and \( T_\Omega^2 \) in the Schatten classes \( S_p \)?

In this note we will concentrate on simply connected domains in \( \mathbb{C}^1 \). To state our main results, we need some definitions and notation.

The Schatten class \( S_p \) \((1 \leq p < \infty)\) consists of all the compact operators \( T \) from a Hilbert space \( H_1 \) to another Hilbert space \( H_2 \) for which the singular numbers \( s_n(T) \) form a sequence belonging to \( l^p \). The singular numbers of the operator \( T \) are the eigenvalues of \( (T^*T)^{1/2} \).

For \( s \in \mathbb{Z}_+ \), \( \alpha > 0 \) and \( 1 \leq p < \infty \), the weighted Sobolev space \( W^{s, \alpha}_p(\mathbb{D}) \) on the unit disk is defined to be the completion of \( C^\infty(\mathbb{D}) \) with respect to the norm
\[
||f||_{W^{s, \alpha}_p(\mathbb{D})} = \left( \sum_{\xi, \eta \in \mathbb{Z}_+} \left( \left( \frac{\partial}{\partial z} \right)^\xi \left( \frac{\partial}{\partial \overline{z}} \right)^\eta f \right)^p \right)^{1/p}.
\]

Note that when \( \alpha = 1 \), \( W^{s, 1}_p(\mathbb{D}) = W^{s}_p(\mathbb{D}) \), which is the usual Sobolev space.

A Jordan curve \( \Gamma \) is of class \( C^{n, \alpha} \) \((n = 1, 2, \cdots, 0 < \alpha < 1)\) if it has a parametrization \( \Gamma : w(t) : 0 \leq t \leq 2\pi \) that is \( n \) times continuously differentiable and satisfies \( w'(t) \neq 0 \) and \( |w^{(n)}(\tau_1) - w^{(n)}(\tau_2)| \leq C|\tau_1 - \tau_2|^\alpha \).

Main Theorem. Let \( \Omega \) be a proper, simply connected domain in \( \mathbb{C}^1 \) and let \( \phi \) be the conformal mapping of \( \mathbb{D} \) onto \( \Omega \). Then:

1. For \( 1 < p < \infty \), if \( \phi' / \overline{\phi'} \in W^{1, \cdot p - 1}_p(\mathbb{D}) \), then \( T_\Omega \in S_p \).
2. For \( p = 1 \), if \( \phi' / \overline{\phi'} \in W^{2, \cdot \gamma, \gamma}(\mathbb{D}) \) for some \( 0 < \gamma < 1 \), then \( T_\Omega \in S_1 \).

From the Main Theorem we have

Corollary A. If \( \Omega \) is a bounded simply connected domain in \( \mathbb{C}^1 \) with \( C^{2, \alpha} \) \((0 < \alpha < 1)\) boundary, then \( T_\Omega \) belongs to all \( S_p \) for \( 1 < p < \infty \).

Corollary B. If \( \Omega \) is a bounded simply connected domain in \( \mathbb{C}^1 \) with \( C^{3, \alpha} \) \((0 < \alpha < 1)\) boundary, then \( T_\Omega \in S_1 \).

For \( T_\Omega^2 \) we have similar results.
2. Compactness

In this section we will give some other conditions for the compactness of the Friedrichs operator. For the bounded simply connected domain case, Friedrichs' compactness theorem [F] will be a corollary of our results.

As we will see in the following lemma, the Friedrichs operator is closely related to the small Hankel operator $h_b$ on the unit disk $\mathbb{D}$. Given $b \in L^2(\mathbb{D})$, the small Hankel operator $h_b$ (with symbol $b$) is defined by

$$h_b f = \overline{P_D(bf)} \quad \text{for} \quad f \in L^2_0(\mathbb{D}).$$

**Lemma 2.1.** Let $\Omega$ be a proper, simply connected domain in $\mathbb{C}^1$ and let $\phi$ be the conformal mapping of $\mathbb{D}$ onto $\Omega$. Then

$$T_\Omega = V_{\phi^{-1}} h_{\phi'/\phi} U_\phi = V_{\phi^{-1}} h_{\overline{P_D(\phi'/\phi)}} U_\phi,$$

where $U_\phi : L^2_0(\Omega) \to L^2_0(\mathbb{D})$, $U_\phi f = f \circ \phi \cdot \phi'$ and $V_{\phi^{-1}} : L^2_0(\mathbb{D}) \to L^2_0(\Omega)$, $V_{\phi^{-1}} g = g \circ \phi^{-1} \cdot (\phi^{-1})'$.

**Proof.** From p. 33 of [B] we have

$$K_\Omega(w, z) = K_D(\phi^{-1}(w), \phi^{-1}(z))((\phi^{-1})'(w)(\phi^{-1})'(z).$$

Then for any $f \in L^2_0(\Omega)$,

$$T_\Omega f(z) = \overline{P_\Omega f(z)} = \int_{\Omega} K_\Omega(w, z)f(w)\,dA(w)$$

$$= \int_{\Omega} K_D(\phi^{-1}(w), \phi^{-1}(z))((\phi^{-1})'(w)(\phi^{-1})'(z)f(w)\,dA(w)$$

$$= \int_{\mathbb{D}} K_D(w, \phi^{-1}(z))\frac{1}{\phi'(w)}((\phi^{-1})'(z)f \circ \phi(w)|\phi'(w)|^2\,dA(w)$$

(by change of variable)

$$= (V_{\phi^{-1}} h_{\phi'/\phi} U_\phi f)(z).$$

Thus

$$T_\Omega = V_{\phi^{-1}} h_{\phi'/\phi} U_\phi.$$  

The second equality follows from the following property of the small Hankel operator:

$$h_b = h_{\overline{P_D b}} \quad \text{for} \quad b \in L^\infty(\mathbb{D}).$$

By using Lemma 2.1, we can get the following compactness result.

**Theorem 2.2.** Let $\Omega$ be a proper, simply connected domain in $\mathbb{C}^1$ and let $\phi$ be the conformal mapping of $\mathbb{D}$ onto $\Omega$. Then $T_\Omega$ is compact if and only if $P_D(\phi'/\phi^*) \in \mathcal{B}_0(\mathbb{D})$, where $\mathcal{B}_0(\mathbb{D})$ is the little Bloch space which consists of holomorphic functions $f$ on $\mathbb{D}$ such that $(1 - |z|^2)f'(z) \to 0$ as $|z| \to 1$.

**Proof.** Since $T_\Omega$ is compact if and only if $\overline{T_\Omega}$ is compact, we only need to consider $\overline{T_\Omega}$. 

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Since \( U_\varphi : L^2_a(\Omega) \to L^2_b(D) \) and \( V_{\varphi^{-1}} : L^2_b(D) \to L^2_a(\Omega) \) are unitary operators, by Lemma 2.1 we have that \( \overline{T}_\Omega \) is compact if and only if the small Hankel operator \( h_{P_b(\varphi'/\varphi')} \) is compact. But by Corollary 2 of Theorem 7.6.6 in [Zh], we know that \( h_{P_b(\varphi'/\varphi')} \) is compact if and only if \( P_b(\varphi'/\overline{\varphi'}) \in \mathcal{B}_0(D) \). This completes the proof. \( \square \)

From Theorem 2.2 we have

**Theorem 2.3.** Let \( \Omega \) be a proper, simply connected domain in \( \mathbb{C}^1 \) and let \( \varphi \) be the conformal mapping of \( D \) onto \( \Omega \). If \( \arg(\varphi') \in C(\overline{D}) \), then \( T_\Omega \) is compact.

**Proof.** If \( \arg(\varphi') \in C(\overline{D}) \), we then have \( \varphi'/\overline{\varphi'} = e^{2i\arg(\varphi')} \in C(\overline{D}) \). By Theorem 5.2.5 of [Zh], \( P_b \) maps \( C(\overline{D}) \) onto \( \mathcal{B}_0(D) \). Thus \( P_b(\varphi'/\overline{\varphi'}) \in \mathcal{B}_0(D) \). Then by Theorem 2.2, \( T_\Omega \) is compact. \( \square \)

As a corollary of Theorem 2.3, we can get the following Friedrichs’ compactness theorem [F].

**Corollary 2.4 ([F]).** If \( \Omega \) is a bounded simply connected domain in \( \mathbb{C}^1 \) with \( C^1 \) boundary, then \( T_\Omega \) is compact.

The proof of Corollary 2.4 follows immediately from Theorem 2.3 and the following lemma.

**Lemma 2.5.** Let \( \varphi \) map \( D \) conformally onto the inner domain of the Jordan curve \( \Gamma \). Then \( \Gamma \) is of class \( C^1 \) if and only if \( \arg(\varphi') \) has a continuous extension to \( \overline{D} \).

**Proof.** See Theorem 3.2 of [Po]. \( \square \)

Given \( b \in L^\infty(D) \), the big Hankel operator \( H_b \) (with symbol \( b \)) is defined by \( H_b f = (I - P_b)(bf) \) for \( f \in L^2_a(D) \). Since the small Hankel operator \( h_b \) is compact whenever the big Hankel operator \( H_b \) is, by using the compactness result of the big Hankel operator \( H_b \), we can also obtain the following sufficient condition for the compactness of the Friedrichs operator \( T_\Omega \).

**Theorem 2.6.** Let \( \Omega \) be a proper, simply connected domain in \( \mathbb{C}^1 \) and let \( \varphi \) be the conformal mapping of \( D \) onto \( \Omega \). If \( \log \varphi' \in \mathcal{B}_0(D) \), then \( T_\Omega \) is compact.

**Proof.** Equivalently, we only need to prove that \( \overline{T}_\Omega \) is compact. By Lemma 2.1, we know that \( \overline{T}_\Omega \) is compact if and only if the small Hankel operator \( h_{\overline{\varphi'}/\varphi'} \) is compact. Now we consider the big Hankel operator \( H_{\overline{\varphi'}/\varphi'} \). Since the symbol function \( \overline{\varphi'}/\varphi' \) is smooth, by Theorem 2 of [Lu] we have that \( H_{\overline{\varphi'}/\varphi'} \) is compact if the following condition is satisfied:

\[
(1 - |z|^2) \left| \frac{\partial}{\partial z} \left( \frac{\varphi'}{\overline{\varphi'}} \right) \right| (z) \to 0 \quad \text{as} \quad |z| \to 1.
\]

But
\[
\left| \frac{\partial}{\partial z} \frac{\overline{\varphi'}}{\varphi'} \right| = \frac{|\overline{\varphi''} \varphi' - \varphi'' \overline{\varphi'}|}{|\varphi'|^2} = \frac{|\varphi''|}{|\varphi'|} = |(\log \varphi')'|.
\]

Therefore condition (2.1) is equivalent to \( \log \varphi' \in \mathcal{B}_0(D) \). Thus, if \( \log \varphi' \in \mathcal{B}_0(D) \), then the big Hankel operator \( H_{\overline{\varphi'}/\varphi'} \) is compact, and this implies that the small Hankel operator \( h_{\overline{\varphi'}/\varphi'} \) is compact. Hence \( \overline{T}_\Omega \) is compact. This completes the proof. \( \square \)
3. Schatten class Friedrichs operators

First, we need to collect a couple of definitions.

For $1 < p < \infty$, the holomorphic Besov space $B_p(\mathbb{D})$ on the unit disk is defined to be the space of holomorphic functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{B_p(\mathbb{D})} = \left( \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z) \right)^{1/p} < \infty.$$  

For $p = 1$, $B_1(\mathbb{D})$ is the space of holomorphic functions $f$ on $\mathbb{D}$ which can be written as

$$f(z) = \sum_{n=1}^{+\infty} a_n \frac{\lambda_n - z}{1 - \lambda_n z}$$

for some sequence $\{a_n\}$ in $l^1$ and $\{\lambda_n\}$ in $\mathbb{D}$.

The weighted holomorphic Sobolev space $W^{s,\gamma}_p(\mathbb{D})$ on the unit disk is the subspace of $W^{s,\alpha}_p(\mathbb{D})$ (defined in Section 1) consisting of holomorphic functions on $\mathbb{D}$.

Now we can consider the membership of $T_\Omega$ in the Schatten classes $S_p$.

**Theorem 3.1.** Let $\Omega$ be a proper, simply connected domain in $\mathbb{C}^1$ and let $\varphi$ be the conformal mapping of $\mathbb{D}$ onto $\Omega$. Then for $1 \leq p < \infty$, $T_\Omega \in S_p$ if and only if $P_\Omega(\varphi'/\varphi') \in B_p(\mathbb{D})$.

**Proof.** Since $U_\varphi : L^2_0(\Omega) \to L^2(\mathbb{D})$ and $V_{\varphi^{-1}} : L^2(\mathbb{D}) \to L^2_0(\Omega)$ are unitary operators, by Lemma 2.1 we have that $T_\Omega \in S_p$ if and only if $h^{-\frac{1}{p}} \in S_p$.

But by Corollary 2 of Theorem 7.6.7 in [Zh], we know that $h^{-\frac{1}{p}} \in S_p$ if and only if $P_\Omega(\varphi'/\varphi') \in B_p(\mathbb{D})$. This completes the proof. $\square$

In order to get our main theorem, we need the following lemma.

**Lemma 3.2.** Let $s \in \mathbb{Z}^+$ and $1 \leq p < \infty$. Then for any $0 < \gamma < p$, $P_\Omega$ projects $W^{s,\gamma}_p(\mathbb{D})$ continuously onto its holomorphic subspace $W^{s,\gamma}_p(\mathbb{D})$.

**Proof.** See [BB]. This lemma is a special case of Corollary 6.3(iii) in [BB]. $\square$

Now we are ready for our main theorem.

**Main Theorem.** Let $\Omega$ be a proper, simply connected domain in $\mathbb{C}^1$ and let $\varphi$ be the conformal mapping of $\mathbb{D}$ onto $\Omega$. Then:

1. For $1 < p < \infty$, if $\varphi'/\varphi^\gamma \in W^{1,\gamma-1}_p(\mathbb{D})$, then $T_\Omega \in S_p$.
2. For $p = 1$, if $\varphi'/\varphi^\gamma \in W^{2,\gamma}_1(\mathbb{D})$ for some $0 < \gamma < 1$, then $T_\Omega \in S_1$.

**Proof.** First we prove (1). For $1 < p < \infty$, using Lemma 3.2, we have

$$\|P_\Omega f\|_{W^{1,\gamma-1}_p(\mathbb{D})} \leq C \|f\|_{W^{1,\gamma-1}_p(\mathbb{D})} \quad \text{for} \quad f \in W^{1,\gamma-1}_p(\mathbb{D}).$$

Then if $\varphi'/\varphi^\gamma \in W^{1,\gamma-1}_p(\mathbb{D})$, we have

$$\|P_\Omega(\varphi'/\varphi^\gamma)\|_{B_p(\mathbb{D})} = \left( \int_{\mathbb{D}} \left| \frac{\partial}{\partial z} (P_\Omega(\varphi'/\varphi^\gamma)) \right|^p (1 - |z|^2)^{p-2} \, dA \right)^{1/p}$$

$$\leq \|P_\Omega(\varphi'/\varphi^\gamma)\|_{W^{1,\gamma-1}_p(\mathbb{D})}$$

$$\leq C \|\varphi'/\varphi^\gamma\|_{W^{1,\gamma-1}_p(\mathbb{D})} < \infty.$$
Thus \( P_B(\phi'/\phi') \in B_p(D) \). It then follows from Theorem 3.1 that \( \overline{T}_\Omega \in S_p \).

Now we prove (2). For \( p = 1 \), using Lemma 3.2, we have for any \( 0 < \gamma < 1 \),

\[
\|P_B f\|_{W^{2,\gamma}_1(D)} \leq C\|f\|_{W^{2,\gamma}_1(D)} \quad \text{for} \quad f \in W^{2,\gamma}_1(D).
\]

On the other hand, it is well known that \( g \in B_1(D) \) if and only if

\[
\int_D |g''(z)| \, dA(z) < \infty.
\]

Then if \( \phi'/\phi' \in W^{2,\gamma}_1(D) \) for some \( 0 < \gamma < 1 \), we have

\[
\int_D \left| \frac{\partial^2}{\partial z^2} \left( P_B(\phi'/\phi') \right) \right| \, dA(z) \leq \int_D \left| \frac{\partial^2}{\partial z^2} \left( P_B(\phi'/\phi') \right) \right| (1 - |z|^2)^{-1} \, dA(z)
\]

\[
\leq \|P_B(\phi'/\phi')\|_{W^{2,\gamma}_1(D)}
\]

\[
\leq C\|\phi'/\phi'\|_{W^{2,\gamma}_1(D)} < \infty.
\]

Thus \( P_B(\phi'/\phi') \in B_1(D) \). It then follows from Theorem 3.1 that \( \overline{T}_\Omega \in S_1 \).

This completes the proof of (2) and of the theorem. \( \square \)

From the Main Theorem we have several corollaries.

**Corollary 3.3.** Let \( \Omega \) be a proper, simply connected domain in \( C^1 \) and let \( \phi \) be the conformal mapping of \( D \) onto \( \Omega \). Then for \( 1 < p < \infty \), if \( \log \phi' \in B_p(D) \), then \( \overline{T}_\Omega \in S_p \).

**Proof.** If \( \log \phi' \in B_p(D) \) with \( 1 < p < \infty \), then

\[
\|\phi'/\phi'\|_{W^{1,p-1}_p(D)}^p = \int_D (1 - |z|^2)^{p-2} |\phi'/\phi'|^p \, dA(z)
\]

\[
+ \int_D (1 - |z|^2)^{p-2} \left| \frac{\partial}{\partial z} (\phi'/\phi') \right|^p \, dA(z)
\]

\[
+ \int_D (1 - |z|^2)^{p-2} \left| \frac{\partial}{\partial z} (\phi'/\phi') \right|^p \, dA(z)
\]

\[
= C + \int_D (1 - |z|^2)^{p-2} \left| \phi''/\phi' \right|^p \, dA(z)
\]

\[
+ \int_D (1 - |z|^2)^{p-2} \left| -\frac{\phi'''}{\phi'} \right|^p \, dA(z)
\]

\[
= C + 2 \int_D (1 - |z|^2)^{p-2} \left| \phi''/\phi' \right|^p \, dA(z)
\]

\[
= C + 2 \|\log \phi'\|_{B_p(D)}^p < \infty.
\]

Thus \( \phi'/\phi' \in W^{1,p-1}_p(D) \). It then follows from (1) of the Main Theorem that \( \overline{T}_\Omega \in S_p \). \( \square \)

**Corollary A.** If \( \Omega \) is a bounded simply connected domain in \( C^1 \) with \( C^{2, \alpha} \) \( 0 < \alpha < 1 \) boundary, then \( \overline{T}_\Omega \) belongs to all \( S_p \) for \( 1 < p < \infty \).

To prove Corollary A, we need the following lemma.
**Lemma 3.4.** Let \( \varphi \) map \( \mathbb{D} \) conformally onto the inner domain of the Jordan curve \( \Gamma \) of class \( C^{n,\alpha} \) where \( n = 1, 2, \ldots \) and \( 0 < \alpha < 1 \). Then \( \varphi^{(n)} \) has a continuous extension to \( \overline{\mathbb{D}} \) and \( \varphi'(z) \neq 0 \) for \( z \in \overline{\mathbb{D}} \).

**Proof.** See Theorems 3.5 and 3.6 of [Po]. \( \square \)

**Proof of Corollary A.** Let \( \varphi \) be the conformal mapping of \( \mathbb{D} \) onto \( \Omega \). By Lemma 3.4, we have that \( \varphi \in C^2(\mathbb{D}) \) and \( \varphi'(z) \neq 0 \) for \( z \in \overline{\mathbb{D}} \). Hence \( \varphi'/\overline{\varphi'} \in C^1(\mathbb{D}) \). It is easy to check that \( C^1(\mathbb{D}) \subset W_p^{1,p-1}(\mathbb{D}) \) \( (1 < p < \infty) \). It then follows from (1) of the Main Theorem that \( T_\Omega \in S_p \) for all \( 1 < p < \infty \).

**Corollary B.** If \( \Omega \) is a bounded simply connected domain in \( \mathbb{C} \) with \( C^{3,\alpha} \) \( (0 < \alpha < 1) \) boundary, then \( \overline{T_\Omega} \in S_1 \).

**Proof.** Let \( \varphi \) be the conformal mapping of \( \mathbb{D} \) onto \( \Omega \). By Lemma 3.4, we have that \( \varphi \in C^3(\mathbb{D}) \) and \( \varphi'(z) \neq 0 \) for \( z \in \overline{\mathbb{D}} \). Hence \( \varphi'/\overline{\varphi'} \in C^2(\mathbb{D}) \). It is easy to see that \( C^2(\mathbb{D}) \subset W_1^{2,\gamma}(\mathbb{D}) \) for any \( 0 < \gamma < 1 \). It then follows from (2) of the Main Theorem that \( \overline{T_\Omega} \in S_1 \). \( \square \)

**4. About \( T_\Omega^2 \)**

As we pointed out in the introduction, \( T_\Omega \) is real-linear but not complex-linear. Sometimes (as in [F] and [N]) it is convenient to study \( T_\Omega^2 \) instead of \( T_\Omega \). \( T_\Omega^2 \) is a complex-linear positive (hence in particular selfadjoint) operator on \( L_2^2(\Omega) \). When \( T_\Omega \) is compact, \( T_\Omega^2 \) has a sequence of eigenvalues \( 1 = \lambda_0 > \lambda_1 \geq \lambda_2 \cdots \) decreasing to zero, and a corresponding sequence of eigenfunctions \( \{\varphi_n(z)\}_{n \geq 0} \) which form an orthonormal basis for \( L_2^2(\Omega) \) and satisfy the remarkable "double orthogonality" relations

\[
\int_\Omega \varphi_m \overline{\varphi}_n dA = 0 = \int_\Omega \varphi_m \varphi_n dA \quad (m \neq n).
\]

See [Sh2] for a more detailed discussion about \( T_\Omega^2 \).

Observe that for \( f \in L_2^2(\Omega) \), \( T_\Omega^2 f = P_\Omega(\overline{T_\Omega f}) = P_\Omega(\overline{T_\Omega f}) \). Hence \( T_\Omega = P_\Omega \overline{T_\Omega} \). Thus \( T_\Omega^2 \in S_p \) whenever \( \overline{T_\Omega} \in S_p \). Then from our main theorem we have

**Theorem 4.1.** Let \( \Omega \) be a proper, simply connected domain in \( \mathbb{C} \) and let \( \varphi \) be the conformal mapping of \( \mathbb{D} \) onto \( \Omega \). Then:

1. For \( 1 < p < \infty \), if \( \varphi'/\overline{\varphi'} \in W_p^{1,p-1}(\mathbb{D}) \), then \( T_\Omega^2 \in S_p \).
2. For \( p = 1 \), if \( \varphi'/\overline{\varphi'} \in W_1^{2,\gamma}(\mathbb{D}) \) for some \( 0 < \gamma < 1 \), then \( T_\Omega^2 \in S_1 \).

Similarly, Corollaries 3.3, A and B are also valid for \( T_\Omega^2 \).

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