ON THE BETTI NUMBER
OF THE IMAGE OF A CODIMENSION- k IMMERSION
WITH NORMAL CROSSINGS

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ABSTRACT. Let $f: M \to N$ be a codimension-$k$ immersion with normal crossings of a closed $m$-dimensional manifold. We show that $f$ is an embedding if and only if the $(m-k+1)$-th Betti numbers of $M$ and $f(M)$ coincide, under a certain condition on the normal bundle of $f$.

1. INTRODUCTION

Let $f: M \to N$ be a codimension-$k$ $C^1$-immersion with normal crossings, where $M$ is a closed $m$-dimensional manifold and $N$ is an $(m+k)$-dimensional manifold ($k \geq 1$). In [BR, BMS1, BMS2], it is shown that when $k = 1$, $H^1(N; Z_2) = 0$, $M$ is orientable, and $f$ is not an embedding, then

$$\beta_0(N - f(M)) \geq 3,$$

where $\beta_i$ denotes the $i$-th Betti number in $Z_2$-coefficient (see also [S]). This is equivalent to showing that $f$ is an embedding if and only if $\beta_m(f(M)) = \beta_m(M)$ (see [BMS2, Lemma 2.2]). In this paper we generalize this result, showing the following.

Theorem 1.1. Let $f: M \to N$ be a codimension-$k$ $C^1$-immersion with normal crossings, where $M$ is a closed $m$-dimensional manifold. Then $f$ is an embedding if and only if

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) \quad \text{and} \quad v_k(f) = w_k(v_f),$$

where $v_k(f) = f^* \circ \beta(\Omega) \in H^k(M; Z_2)$, $\Omega \in H^k(N, N - f(M); Z_2)$ is the transverse class defined in [He], $\beta: H^k(N, N - f(M); Z_2) \to H^k(N; Z_2)$ is the homomorphism induced by the inclusion, and $w_k(v_f) \in H^k(M; Z_2)$ is the top Stiefel-Whitney class of the normal bundle $v_f$ of the immersion $f$.

Note that, when $k = 1$, $H^1(N; Z_2) = 0$, and $M$ is orientable, we always have $v_k(f) = w_k(v_f) = 0$.

In particular, we have the following.

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Corollary 1.2. Let $f: M \to N$ be a codimension-$k$ $C^1$-immersion with normal crossings, where $M$ is a closed $m$-dimensional manifold. Suppose that either $f^*: H^k(N; \mathbb{Z}_2) \to H^k(M; \mathbb{Z}_2)$ or $f_*: H_m(M; \mathbb{Z}) \to H_m(N; \mathbb{Z}_2)$ is the zero map. Then $f$ is an embedding if and only if
\[ \beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) \quad \text{and} \quad w_k(\nu_f) = 0. \]

Note that the top Stiefel-Whitney class $\omega_k(\nu_f)$ of the normal bundle $\nu_f$ is the modulo 2 reduction of the Euler class, which is the obstruction to the existence of a nowhere zero cross section. Note also that $w_k(\nu_f)$ depends only on the homotopy class of $f$.

As to the Betti number of the complement of the image of an immersion, we have the following.

Corollary 1.3. Let $f: M \to N$ be a codimension-$k$ $C^1$-immersion with normal crossings, where $M$ is a closed $m$-dimensional manifold, $\beta_k(N) = \beta_{2k-1}(N) = \beta_{2k-2}(N) = 0$, and $w_k(\nu_f) = 0$. Here $\beta_{2k-2}$ denotes the dimension of the reduced $(2k-2)$-th homology group in $\mathbb{Z}_2$-coefficient. Then $f$ is an embedding if and only if $\beta_{2k-2}(N - f(M)) = \beta_{k-1}(M)$.

Note that Corollary 1.3 is a generalization of the results in [BR, BMS1, BMS2] concerning the case $k = 1$ (see also [S]).

In the following, all the homology and cohomology groups are with $\mathbb{Z}_2$-coefficients.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Set $A = \{x \in M: f^{-1}(f(x)) \neq \{x\}\}$ and $B = f(A)$. Note that $A$ and $B$ are ANR. We suppose that $f$ is not an embedding; i.e., $A \neq \emptyset$. Consider the following diagram of homologies with exact rows:

\[
\begin{array}{ccccccccc}
\cdots & \to & H_i(A) & \to & H_i(M) & \to & H_i(M, A) & \to & H_{i-1}(A) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & H_i(B) & \to & H_i(f(M)) & \to & H_i(f(M), B) & \to & H_{i-1}(B) & \to & \cdots \\
\end{array}
\]

where the vertical homomorphisms are induced by $f$. Note that the homomorphism $f_*: H_i(M, A) \to H_i(f(M), B)$ is an isomorphism by excision. Then it is not difficult to extract the following exact sequence:

\[
\cdots \to H_{m-k+1}(A) \to H_{m-k+1}(B) \oplus H_{m-k+1}(M) \to H_{m-k+1}(f(M)) \to H_{m-k}(A) \to H_{m-k}(B) \oplus H_{m-k}(M) \to \cdots.
\]

Since $A$ and $B$ are of dimension $m - k$, we have the exact sequence

\[
0 \to H_{m-k+1}(M) \to H_{m-k+1}(f(M)) \to H_{m-k}(A) \to H_{m-k}(B) \oplus H_{m-k}(M) \to \cdots,
\]

where $\alpha = (f|A)_* \oplus j_*$ and $j: A \to M$ is the inclusion map. Then we have

\[
\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) + \dim \ker \alpha.
\]

Now consider the fundamental class $[A] \in H_{m-k}(A)$, which is known to exist and to be non-zero ([He]). Then we have $(f|A)_*[A] = 0$, since $f|A$ is a double cover away from the codimension-$k$ set $\{x \in A: \#(f^{-1}(f(x))) \geq 3\}$, where $\#$
denotes the cardinality. On the other hand, by Herbert [He], we have a formula for calculating $j_*[A]$, which is

$$j_*[A] = D_M \circ f^* \circ \beta(\Omega_1) - D_M(w_k(\nu_f)) = D_M(v_k(f) - w_k(\nu_f)),$$

where $D_M: H^k(M) \to H_{m-k}(M)$ is the Poincaré dual, $\beta: H^k(N, N-f(M)) \to H^k(N)$ is the homomorphism induced by the inclusion $(N, \emptyset) \to (N, N-f(M))$, and $\Omega_1 \in H^k(N, N-f(M))$ is the transverse class defined in [He]. (For this formula, see also [W, (18.5)].) Now suppose that $v_k(f) = w_k(\nu_f)$. Then we have $j_*[A] = 0$. This implies that

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) + \dim \ker \alpha > \beta_{m-k+1}(M),$$

since $[A]$ is a non-zero element of $\ker \alpha$. Thus, if $f$ is not an embedding, we have

$$\beta_{m-k+1}(f(M)) > \beta_{m-k+1}(M)$$

or

$$v_k(f) \neq w_k(\nu_f).$$

On the other hand, if $f$ is an embedding, we clearly have

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M).$$

Furthermore, the formula of Herbert [He] cited above shows $v_k(f) = w_k(\nu_f)$. This completes the proof of Theorem 1.1  

Proof of Corollary 1.2. If $f^*: H^k(N) \to H^k(M)$ is the zero map, it is easy to see that $v_k(f) = 0$, by the definition of $v_k(f)$. Thus, for the proof of Corollary 1.2, we have only to show that, if $f_*: H_m(M) \to H_m(N)$ is the zero map, then $v_k(f) = 0$. Let

$$D_1: H^k(N, N-f(M)) \to H_m(f(M))$$

and

$$D_2: H_m(N, N-f(M)) \to H^k(f(M))$$

be the duality isomorphisms. Furthermore, we denote by $i: f(M) \to N$ and $l: (N, \emptyset) \to (N, N-f(M))$ the inclusion maps. First note that $D_1(\Omega_1) = f_*[M] \in H_m(f(M))$ by [He, Proposition 4.1], where $[M] \in H_m(M)$ is the fundamental class of $M$. Then we have

$$v_k(f) = f^* \circ \beta(\Omega_1) = f^* \circ (l \circ i)^*(\Omega_1) = f^* \circ D_2 \circ (l \circ i)_* \circ D_1(\Omega_1) = f^* \circ D_2 \circ (l \circ i)_*(f_*[M]) = f^* \circ D_2 \circ l_*(f_*[M]) = 0,$$

since $f_*[M] = 0 \in H_m(N)$ by our hypothesis. This completes the proof.  

Remark 2.1. We can interpret the cohomology classes $v_k(f)$ and $w_k(\nu_f) \in H^k(M)$ geometrically as follows. Let $\gamma \in H_k(M)$ be an arbitrary homology class and $C (\subset M)$ a singular cycle representing $\gamma$. Then we see that $\langle v_k(f), \gamma \rangle$ is equal to the modulo 2 intersection number of $f(M)$ and $f(C)$ in $N$, where we move $f(C)$ slightly so that it intersects $f(M)$ transversely. On the other hand, $\langle w_k(\nu_f), \gamma \rangle$ is equal to the modulo 2 self-intersection number of $C$ in
Figure 1

the total space of $i^*\nu_f$, where $i: C \rightarrow M$ is the inclusion map. In other words, 
\[ \langle w_k(\nu_f), \gamma \rangle \] is equal to the modulo 2 intersection number of $f(M)$ and $f(C)$ in $N$ off the self-intersection of $f$.

**Remark 2.2.** The condition about the top Stiefel-Whitney class of the normal bundle $\nu_f$ of $f$ is necessary in Theorem 1.1 and Corollaries 1.2 and 1.3. For example, consider the immersion with normal crossings $f: K \rightarrow \mathbb{R}^3$ as in Figure 1, where $K$ is the Klein bottle. We see that the immersion $f$ is not an embedding, but that 
\[ \beta_2(f(K)) = \beta_1(K) = 1. \]

Note that, in this example, we have $0 = v_1(f) \neq w_1(\nu_f)$. Furthermore, Theorem 1.1 implies that, for any immersion $g: K \rightarrow N$ with normal crossings into a 3-manifold $N$ with $g(K)$ homeomorphic to $f(K)$, $v_1(g)$ never coincides with $w_1(\nu_g)$.

**Remark 2.3.** Suppose that there exist integers $p$ and $q$ such that $q \leq p + 1$, $p + q = k$, $f^*(w_i(N)) = 0$ ($0 < i < q$), and $w_j(M) = 0$ ($0 < j \leq p$), where $w_i$ denotes the $i$-th Stiefel-Whitney class. Then we have $f^*(w_k(N)) = w_k(M) + w_k(\nu_f)$. This can be proved as follows. By the definition of the normal bundle of an immersion, we have 
\[ f^*(w(N)) = w(M) \cup w(\nu_f), \]
where $w$ denotes the total Stiefel-Whitney class. Then we have 
\[ w(\nu_f) = f^*(w(N)) \cup (w(M))^{-1}, \]
which implies that $w_i(\nu_f) = 0$ for $0 < i < q$. Then we have 
\[ f^*(w_k(N)) = w_k(M) + w_k(\nu_f). \]

Thus, if in addition we have $w_k(M) = f^*(w_k(N)) = 0$, then we have $w_k(\nu_f) = 0$. For example, if $k = 1$ and $M$ and $N$ are orientable, $w_1(\nu_f)$ always vanishes. If $k = 2$ and $M$ and $N$ are spin manifolds, $w_2(\nu_f)$ always vanishes.

**Proof of Corollary 1.3.** First note that, since $H_k(N) = 0$, the hypotheses of Corollary 1.2 are satisfied for $f$. Now consider the following exact sequence of homology:
\[ \tilde{H}_{2k-1}(N) \rightarrow H_{2k-1}(N, N - f(M)) \rightarrow \tilde{H}_{2k-2}(N - f(M)) \rightarrow \tilde{H}_{2k-2}(N). \]
Note that
\[ \tilde{H}_{2k-1}(N) = 0, \quad H_{2k-1}(N, N - f(M)) \cong H^{m-k+1}(f(M)), \]
and
\[ \tilde{H}_{2k-2}(N) = 0. \]
Thus we have
\[ \tilde{\beta}_{2k-2}(N - f(M)) = \beta_{m-k+1}(f(M)). \]
Note also that \( \beta_{m-k+1}(M) = \beta_{k-1}(M) \) by Poincaré duality. Then, combining this with Corollary 1.2, we obtain the conclusion. This completes the proof. \( \square \)

**Remark 2.4.** Corollary 1.3 seems a little bit tedious. However, when \( N = \mathbb{R}^{m+k} \), it takes a simpler form as follows: a codimension- \( k \) \( C^1 \)-immersion with normal crossings \( f: M \to \mathbb{R}^{m+k} \) of a closed \( m \)-dimensional manifold \( M \) with vanishing \( k \)-th dual Stiefel-Whitney class \( \bar{w}_k(M) \) (\( \in H^k(M) \)) is an embedding if and only if \( \tilde{\beta}_{2k-2}(\mathbb{R}^{m+k} - f(M)) = \beta_{k-1}(M) \).

**Remark 2.5.** In [Hi], Hirsch has shown that, if \( f: M \to N \) is a codimension- \( k \) proper \( C^2 \)-immersion and \( H_k(N) = 0 \), then \( H_{k-1}(N - f(M)) \) is non-trivial. Using the techniques used in the proof of Theorem 1.1, we can prove a refinement of Hirsch’s result for immersions with normal crossings as follows. Let \( f: M \to N \) be a codimension- \( k \) \( C^1 \)-immersion with normal crossings, where \( M \) is a closed \( m \)-dimensional manifold, \( \dim H_{k-1}(N) \) is finite, and \( H_k(N) = 0 \). Then we have the following.

1. We always have
   \[ \beta_{k-1}(N - f(M)) (= \beta_{k-1}(N) + \beta_m(f(M))) \geq \beta_{k-1}(N) + \beta_0(M). \]

2. When \( k = 1 \) and \( w_k(M) = 0 \), the equality holds in (1) if and only if \( f \) is an embedding.

3. When \( k \geq 2 \), the equality in (1) always holds.

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**References**


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