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ON THE BETTI NUMBER OF THE IMAGE OF A CODIMENSION-k IMMERSION WITH NORMAL CROSSINGS

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ABSTRACT. Let $f\colon M \to N$ be a codimension-k immersion with normal crossings of a closed m-dimensional manifold. We show that f is an embedding if and only if the $(m-k+1)$-th Betti numbers of M and $f(M)$ coincide, under a certain condition on the normal bundle of f.

1. INTRODUCTION

Let $f\colon M \to N$ be a codimension-k C^1-immersion with normal crossings, where M is a closed m-dimensional manifold and N is an $(m+k)$-dimensional manifold $(k \geq 1)$. In [BR, BMS1, BMS2], it is shown that when $k = 1$, $H^1(N; \mathbf{Z}_2) = 0$, M is orientable, and f is not an embedding, then

$$\beta_0(N - f(M)) \geq 3,$$

where β_i denotes the i-th Betti number in $\mathbf{Z}_2$-coefficient (see also [S]). This is equivalent to showing that f is an embedding if and only if $\beta_m(f(M)) = \beta_m(M)$ (see [BMS2, Lemma 2.2]). In this paper we generalize this result, showing the following.

Theorem 1.1. *Let $f\colon M \to N$ be a codimension-k C^1-immersion with normal crossings, where M is a closed m-dimensional manifold. Then f is an embedding if and only if*

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) \quad \text{and} \quad v_k(f) = w_k(\nu_f),$$

where $v_k(f) = f^* \circ \beta(\Omega_1) \in H^k(M; \mathbf{Z}_2)$, $\Omega_1 \in H^k(N, N - f(M); \mathbf{Z}_2)$ is the transverse class defined in [He], *$\beta\colon H^k(N, N - f(M); \mathbf{Z}_2) \to H^k(N; \mathbf{Z}_2)$ is the homomorphism induced by the inclusion, and $w_k(\nu_f) \in H^k(M; \mathbf{Z}_2)$ is the top Stiefel-Whitney class of the normal bundle ν_f of the immersion f.*

Note that, when $k = 1$, $H^1(N; \mathbf{Z}_2) = 0$, and M is orientable, we always have $v_k(f) = w_k(\nu_f) = 0$.

In particular, we have the following.

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Corollary 1.2. *Let $f: M \to N$ be a codimension-k C^1-immersion with normal crossings, where M is a closed m-dimensional manifold. Suppose that either $f^*: H^k(N; \mathbf{Z}_2) \to H^k(M; \mathbf{Z}_2)$ or $f_*: H_m(M; \mathbf{Z}) \to H_m(N; \mathbf{Z}_2)$ is the zero map. Then f is an embedding if and only if*

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) \quad \text{and} \quad w_k(\nu_f) = 0.$$

Note that the top Stiefel-Whitney class $\omega_k(\nu_f)$ of the normal bundle ν_f is the modulo 2 reduction of the Euler class, which is the obstruction to the existence of a nowhere zero cross section. Note also that $w_k(\nu_f)$ depends only on the homotopy class of f.

As to the Betti number of the complement of the image of an immersion, we have the following.

Corollary 1.3. *Let $f: M \to N$ be a codimension-k C^1-immersion with normal crossings, where M is a closed m-dimensional manifold, $\beta_k(N) = \beta_{2k-1}(N) = \tilde{\beta}_{2k-2}(N) = 0$, and $w_k(\nu_f) = 0$. Here $\tilde{\beta}_{2k-2}$ denotes the dimension of the reduced $(2k-2)$-th homology group in $\mathbf{Z}_2$-coefficient. Then f is an embedding if and only if $\tilde{\beta}_{2k-2}(N - f(M)) = \beta_{k-1}(M)$.*

Note that Corollary 1.3 is a generalization of the results in [BR, BMS1, BMS2] concerning the case $k = 1$ (see also [S]).

In the following, all the homology and cohomology groups are with $\mathbf{Z}_2$-coefficients.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Set $A = \{x \in M: f^{-1}(f(x)) \neq \{x\}\}$ and $B = f(A)$. Note that A and B are ANR. We suppose that f is not an embedding; i.e., $A \neq \varnothing$. Consider the following diagram of homologies with exact rows:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_i(A) & \longrightarrow & H_i(M) & \longrightarrow & H_i(M, A) & \longrightarrow & H_{i-1}(A) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H_i(B) & \longrightarrow & H_i(f(M)) & \longrightarrow & H_i(f(M), B) & \longrightarrow & H_{i-1}(B) & \longrightarrow \cdots
\end{array}$$

where the vertical homomorphisms are induced by f. Note that the homomorphism $f_*: H_i(M, A) \to H_i(f(M), B)$ is an isomorphism by excision. Then it is not difficult to extract the following exact sequence:

$$\begin{aligned}
\cdots \to H_{m-k+1}(A) \to H_{m-k+1}(B) \oplus H_{m-k+1}(M) \to H_{m-k+1}(f(M)) \\
\to H_{m-k}(A) \to H_{m-k}(B) \oplus H_{m-k}(M) \to \cdots .
\end{aligned}$$

Since A and B are of dimension $m-k$, we have the exact sequence

$$\begin{aligned}
0 \to H_{m-k+1}(M) \to H_{m-k+1}(f(M)) \\
\to H_{m-k}(A) \xrightarrow{\alpha} H_{m-k}(B) \oplus H_{m-k}(M) \to \cdots ,
\end{aligned}$$

where $\alpha = (f|A)_* \oplus j_*$ and $j: A \to M$ is the inclusion map. Then we have

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) + \dim \ker \alpha.$$

Now consider the fundamental class $[A] \in H_{m-k}(A)$, which is known to exist and to be non-zero ([He]). Then we have $(f|A)_*[A] = 0$, since $f|A$ is a double cover away from the codimension-k set $\{x \in A: \sharp(f^{-1}(f(x))) \geq 3\}$, where $\sharp$

denotes the cardinality. On the other hand, by Herbert [He], we have a formula for calculating $j_*[A]$, which is

$$\begin{aligned} j_*[A] &= D_M \circ f^* \circ \beta(\Omega_1) - D_M(w_k(\nu_f)) \\ &= D_M(v_k(f) - w_k(\nu_f)), \end{aligned}$$

where $D_M\colon H^k(M) \to H_{m-k}(M)$ is the Poincaré dual, $\beta\colon H^k(N, N-f(M)) \to H^k(N)$ is the homomorphism induced by the inclusion $(N, \varnothing) \to (N, N-f(M))$, and $\Omega_1 \in H^k(N, N-f(M))$ is the transverse class defined in [He]. (For this formula, see also [W, (18.5)].) Now suppose that $v_k(f) = w_k(\nu_f)$. Then we have $j_*[A] = 0$. This implies that

$$\begin{aligned} \beta_{m-k+1}(f(M)) &= \beta_{m-k+1}(M) + \dim\ker\alpha \\ &> \beta_{m-k+1}(M), \end{aligned}$$

since $[A]$ is a non-zero element of $\ker\alpha$. Thus, if f is not an embedding, we have

$$\beta_{m-k+1}(f(M)) > \beta_{m-k+1}(M)$$

or

$$v_k(f) \neq w_k(\nu_f).$$

On the other hand, if f is an embedding, we clearly have

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M).$$

Furthermore, the formula of Herbert [He] cited above shows $v_k(f) = w_k(\nu_f)$. This completes the proof of Theorem 1.1 □

Proof of Corollary 1.2. If $f^*\colon H^k(N) \to H^k(M)$ is the zero map, it is easy to see that $v_k(f) = 0$, by the definition of $v_k(f)$. Thus, for the proof of Corollary 1.2, we have only to show that, if $f_*\colon H_m(M) \to H_m(N)$ is the zero map, then $v_k(f) = 0$. Let

$$D_1\colon H^k(N, N-f(M)) \to H_m(f(M))$$

and

$$D_2\colon H_m(N, N-f(M)) \to H^k(f(M))$$

be the duality isomorphisms. Furthermore, we denote by $i\colon f(M) \to N$ and $l\colon (N, \varnothing) \to (N, N-f(M))$ the inclusion maps. First note that $D_1(\Omega_1) = f_*[M] \in H_m(f(M))$ by [He, Proposition 4.1], where $[M] \in H_m(M)$ is the fundamental class of M. Then we have

$$\begin{aligned} v_k(f) &= f^* \circ \beta(\Omega_1) = f^* \circ (l \circ i)^*(\Omega_1) \\ &= f^* \circ D_2 \circ (l \circ i)_* \circ D_1(\Omega) = f^* \circ D_2 \circ (l \circ i)_*(f_*[M]) \\ &= f^* \circ D_2 \circ l_*(f_*[M]) = 0, \end{aligned}$$

since $f_*[M] = 0 \in H_m(N)$ by our hypothesis. This completes the proof. □

Remark 2.1. We can interpret the cohomology classes $v_k(f)$ and $w_k(\nu_f) \in H^k(M)$ geometrically as follows. Let $\gamma \in H_k(M)$ be an arbitrary homology class and C $(\subset M)$ a singular cycle representing γ. Then we see that $\langle v_k(f), \gamma\rangle$ is equal to the modulo 2 intersection number of $f(M)$ and $f(C)$ in N, where we move $f(C)$ slightly so that it intersects $f(M)$ transversely. On the other hand, $\langle w_k(\nu_f), \gamma\rangle$ is equal to the modulo 2 self-intersection number of C in

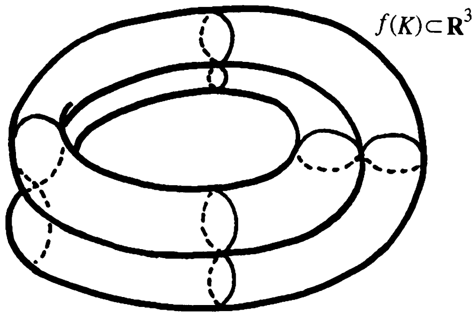

FIGURE 1

the total space of $i^*\nu_f$, where $i: C \to M$ is the inclusion map. In other words, $\langle w_k(\nu_f), \gamma \rangle$ is equal to the modulo 2 intersection number of $f(M)$ and $f(C)$ in N off the self-intersection of f.

Remark 2.2. The condition about the top Stiefel-Whitney class of the normal bundle ν_f of f is necessary in Theorem 1.1 and Corollaries 1.2 and 1.3. For example, consider the immersion with normal crossings $f: K \to \mathbf{R}^3$ as in Figure 1, where K is the Klein bottle. We see that the immersion f is not an embedding, but that

$$\beta_2(f(K)) = \beta_1(K) = 1.$$

Note that, in this example, we have $0 = v_1(f) \neq w_1(\nu_f)$. Furthermore, Theorem 1.1 implies that, for any immersion $g: K \to N$ with normal crossings into a 3-manifold N with $g(K)$ homeomorphic to $f(K)$, $v_1(g)$ never coincides with $w_1(\nu_g)$.

Remark 2.3. Suppose that there exist integers p and q such that $q \leq p+1$, $p+q = k$, $f^*(w_i(N)) = 0$ $(0 <^\forall i < q)$, and $w_j(M) = 0$ $(0 <^\forall j \leq p)$, where w_i denotes the i-th Stiefel-Whitney class. Then we have $f^*(w_k(N)) = w_k(M) + w_k(\nu_f)$. This can be proved as follows. By the definition of the normal bundle of an immersion, we have

$$f^*(w(N)) = w(M) \cup w(\nu_f),$$

where w denotes the total Stiefel-Whitney class. Then we have

$$w(\nu_f) = f^*(w(N)) \cup (w(M))^{-1},$$

which implies that $w_i(\nu_f) = 0$ for $0 <^\forall i < q$. Then we have

$$f^*(w_k(N)) = w_k(M) + w_k(\nu_f).$$

Thus, if in addition we have $w_k(M) = f^*(w_k(N)) = 0$, then we have $w_k(\nu_f) = 0$. For example, if $k = 1$ and M and N are orientable, $w_1(\nu_f)$ always vanishes. If $k = 2$ and M and N are spin manifolds, $w_2(\nu_f)$ always vanishes.

Proof of Corollary 1.3. First note that, since $H_k(N) = 0$, the hypotheses of Corollary 1.2 are satisfied for f. Now consider the following exact sequence of homology:

$$\widetilde{H}_{2k-1}(N) \to H_{2k-1}(N, N - f(M)) \to \widetilde{H}_{2k-2}(N - f(M)) \to \widetilde{H}_{2k-2}(N).$$

Note that

$$\widetilde{H}_{2k-1}(N) = 0, \qquad H_{2k-1}(N, N - f(M)) \cong H^{m-k+1}(f(M)),$$

and

$$\widetilde{H}_{2k-2}(N) = 0.$$

Thus we have

$$\tilde{\beta}_{2k-2}(N - f(M)) = \beta_{m-k+1}(f(M)).$$

Note also that $\beta_{m-k+1}(M) = \beta_{k-1}(M)$ by Poincaré duality. Then, combining this with Corollary 1.2, we obtain the conclusion. This completes the proof. □

Remark 2.4. Corollary 1.3 seems a little bit tedious. However, when $N = \mathbf{R}^{m+k}$, it takes a simpler form as follows: a codimension-k C^1-immersion with normal crossings $f: M \to \mathbf{R}^{m+k}$ of a closed m-dimensional manifold M with vanishing k-th dual Stiefel-Whitney class $\overline{w}_k(M)$ ($\in H^k(M)$) is an embedding if and only if $\tilde{\beta}_{2k-2}(\mathbf{R}^{m+k} - f(M)) = \beta_{k-1}(M)$.

Remark 2.5. In [Hi], Hirsch has shown that, if $f: M \to N$ is a codimension-k proper C^2-immersion and $H_k(N) = 0$, then $H_{k-1}(N - f(M))$ is non-trivial. Using the techniques used in the proof of Theorem 1.1, we can prove a refinement of Hirsch's result for immersions with normal crossings as follows. Let $f: M \to N$ be a codimension-k C^1-immersion with normal crossings, where M is a closed m-dimensional manifold, $\dim H_{k-1}(N)$ is finite, and $H_k(N) = 0$. Then we have the following.

(1) We always have

$$\begin{aligned}\beta_{k-1}(N - f(M)) \ (= \beta_{k-1}(N) + \beta_m(f(M))) \\ \geq \beta_{k-1}(N) + \beta_0(M).\end{aligned}$$

(2) When $k = 1$ and $w_k(M) = 0$, the equality holds in (1) if and only if f is an embedding.

(3) When $k \geq 2$, the equality in (1) always holds.

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