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PROJECTIONS ALGEBRAICALLY GENERATE THE BOUNDED OPERATORS ON REAL OR QUATERNIONIC HILBERT SPACE

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ABSTRACT. We prove the theorem of the title.

Let H be an infinite-dimensional, separable Hilbert space with scalars the real numbers $\mathbb{R}$ or the quaternions $\mathbb{H}$. Consider $\mathscr{B}(H)$, the $*$-ring of all bounded, everywhere-defined linear operators T on H where linear means that $T(\rho x) = \rho T(x)$ for every $x \in H$ and every $\rho \in \mathbb{R}$, resp. every $\rho \in \mathbb{H}$. In both cases $\mathscr{B}(H)$ is an algebra over $\mathbb{R}$, and our purpose here is to establish that every $T \in \mathscr{B}(H)$ equals a finite sum $\sum \rho_i M_i$ where $\rho_i \in \mathbb{R}$ and each M_i is a finite product of projections. ($E \in \mathscr{B}(H)$ is a projection when $E^2 = E^* = E$.) For complex Hilbert space the theorem is known [D, Proposition 7; F-T]. Our proof actually covers all three cases.

Every nonnegative $T \in \mathscr{B}(H)$ has a unique nonnegative square root, $T^{1/2}$, that commutes with every $A \in \mathscr{B}(H)$ that commutes with T. The proof of this result presented in [R-S, §104], ostensibly just for real and complex H, applies without change to the quaternionic case. Using this result, we directly deduce polar decomposition: every $T \in \mathscr{B}(H)$ can be written $T = W(T^*T)^{1/2}$ where the partial isometry W satisfies $\ker(W) = \ker(T)$. In particular, applying polar decomposition to a skew operator $S = -S^*$, we get $S = WT$ where $T = (-S^2)^{1/2} \geq 0$ and $W^*W = WW^* = E$, the projection on $\ker(S)^\perp = \operatorname{im}(S)^{\perp\perp}$. Then $U = W + (I - E)$ is unitary, $U^*U = UU^* = I$, and $S = UT$.

Our proof rests on the same matrix representation device used so efficiently by Fillmore and Topping in the complex case [F-T]; a few modifications are needed to get round the fact that $\mathbb{R}$ and $\mathbb{H}$ contain no central skew elements. $\mathscr{B}(H)$ is isomorphic as a real $*$-algebra to the $*$-algebra $A = M_2(\mathscr{B}(K))$ of all 2×2 matrices with entries from $\mathscr{B}(K)$, where K is another infinite-dimensional, separable Hilbert space, correspondingly real or quaternionic. The $*$-operation in A is $*$-transpose. So we may prove our theorem for A. Let $\overset{\circ}{A}$ stand for the real $*$-subalgebra of A generated algebraically by its projections. We prove $\overset{\circ}{A} = A$.

Given $T \in \mathscr{B}(K)$, $0 \leq T \leq I$, and given unitary $U \in \mathscr{B}(K)$, one checks

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easily that both

$$P(T) = \begin{bmatrix} T & (T(I-T))^{1/2} \\ (T(I-T))^{1/2} & I-T \end{bmatrix} \quad \text{and} \quad Q(U) = \frac{1}{2}\begin{bmatrix} I & U \\ U^* & I \end{bmatrix}$$

are projections in A, thus lie in $\overset{\circ}{A}$. Hence the matrices $P(I)$, $2Q(U)-I$, and $2(P(I)Q(U)-Q(U)P(I))$ all lie in $\overset{\circ}{A}$. Using these matrices and noting that if $T = T^* \in \mathscr{B}(K)$, then $0 \leq \rho T + \sigma I \leq I$ for suitable positive real numbers ρ and σ, one establishes that if $\overset{\circ}{A}$ contains every

$$\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

where $T \in \mathscr{B}(K)$ satisfies $0 \leq T \leq I$ and $S \in \mathscr{B}(K)$ satisfies $S^* = -S$, then $\overset{\circ}{A} = A$. The first matrix is $P(I)P(T)P(I)$, so lies in $\overset{\circ}{A}$. For the second, write $S = UT$, $UU^* = U^*U = I$, $T \geq 0$. Then the second matrix is the product of two elements of $\overset{\circ}{A}$,

$$\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix},$$

so also lies in $\overset{\circ}{A}$. Thus $\overset{\circ}{A} = A$.

In the complex case, Fillmore has proved the much stronger result that every $T \in \mathscr{B}(H)$ is a linear combination of 17 projections [F]. Of course this cannot be true in the real and quaternionic cases, as every real-linear combination of projections is selfadjoint.

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