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REMARKS ABOUT γ-SETS AND BOREL-DENSE SETS

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ABSTRACT. We show, assuming Martin's Axiom, that every set of cardinality the continuum containing a Borel-dense set of cardinality less than the continuum is a γ-set but is not a hereditarily γ-set. This answers a question of D. H. Fremlin and J. Jasinski.

A family $J \subset P(X)$ is an ω-cover of X if for every finite set $F \subset X$ there exists $B \in J$ such that $F \subset B$. A topological space X is a γ-set if for every open ω-cover J of X there exists a family $\{D_n: n \in \omega\} \subset J$ such that $X \subset \bigcup_m \bigcap_{n>m} D_n$.

This definition was introduced by Gerlits and Nagy [3]. They showed that a space X is a γ-set iff $C(X)$ is a Frechet space. Galvin and Miller [2] showed that under MA there is a γ-set of the reals of size continuum. Since every γ-set of reals is strong measure zero (see [3]), by Laver's result (see [4]) it is consistent that there is no uncountable γ-set of reals. Under $\diamondsuit_{\omega_1}$ Todorcevic (see [2]) showed that there is a hereditarily γ-set of reals of size $\mathbf{c}$. It is still an open problem whether MA or CH implies the existence of a hereditarily γ-set of reals of size $\mathbf{c}$.

A subset A of separable metric space X is Borel-dense in X if A meets every Borel subset of cardinality continuum of X.

D. H. Fremlin and J. Jasinski [1] showed that if Martin's Axiom holds and there exists $\kappa < \mathbf{c}$ such that $P(\kappa)$ contains a proper, uniform, ω_1-saturated, κ-additive ideal, then there exists a set X of reals of cardinality the continuum containing a subset D of cardinality less than the continuum Borel-dense in X. They also showed that X can be a γ-set. They asked a question: Can X be a hereditarily γ-set?

Theorem. *Assume Martin's Axiom. If $X \subset \mathbf{R}$ of cardinality continuum contains a subset D of cardinality less than continuum Borel-dense in X, then X is a γ-set but is not a hereditarily γ-set.*

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Lemma 1. *Assume Martin's Axiom. Let $X \subset \mathbf{R}$. If there exists $D \subset X$ such that $|D| < \mathbf{c}$ and for every Borel subset B containing D, $|X \backslash B| < \mathbf{c}$, then X is a γ-set.*

Proof. Let J be an open ω-cover of X. We will show that there exists a family $\{B_n^k : n, k \in \omega\} \subset J$ such that for every $Y \in [X]^{<\omega}$ there exists $k \in \omega$ such that for infinitely many n, $Y \subset B_n^k$ and for every $k \in \omega$, $D \subset \bigcup_m \bigcap_{n>m} B_n^k$. We know that every set of reals of cardinality less than continuum is a γ-set (see [2]). There exists a family $\{B_n^0 : n \in \omega\} \subset J$ such that $D \subset \bigcup_m \bigcap_{n>m} B_n^0$. Let $X_1 = X \backslash \bigcup_m \bigcap_{n>m} B_n^0$. Then $|X_1| < \mathbf{c}$. Let $X_1, X_2, \dots, X_k \subset X$ be such that $|X_i| < \mathbf{c}$ for $1 \le i \le k$. Then $\bigcup_{i=1}^k X_i \cup D$ is a γ-set. Thus there exists a family $\{B_n^k : n \in \omega\} \subset J$ such that $\bigcup_{i=1}^k X_i \cup D \subset \bigcup_m \bigcap_{n>m} B_n^k$, and we let $X_{k+1} = X \backslash \bigcup_m \bigcap_{n>m} B_n^k$. It is not hard to see that the family $\{B_n^k : n, k \in \omega\}$ has the required properties.

For $d \in D$ there exists a function $f_d : \omega \to \omega$ such that if $f_d(k) = m$, then $d \in \bigcap_{n \ge m} B_n^k$. Since $|D| < \mathbf{c}$, there exists a function $g : \omega \to \omega$ such that $f_d <^* g$. Let $J_0 = \{B_n^k : n > g(k) \text{ and } k, n \in \omega\} = \{C_n : n \in \omega\}$. Observe that $D \subset \bigcup_m \bigcap_{n>m} C_n$. Because for every $d \in D$ there exists $l \in \omega$ such that if $k > l$ and $n > g(k)$, then $d \in B_n^k$, if $k \le l$, then there exist only finitely many n such that $d \notin B_n^k$. Thus d must be in all but finitely many C_n. We know also that every finite subset of X is contained in infinitely many C_n. Let $Z = X \backslash \bigcup_m \bigcap_{n>m} C_n$. Evidently $|Z| < \mathbf{c}$. If $|Z| < \omega$, then there exists an increasing sequence n_k such that $Z \subset C_{n_k}$ for every k. Thus $X \subset \bigcup_m \bigcap_{k>m} C_{n_k}$. If $|Z| \ge \omega$, then $\{C_n \backslash \{y_n\} : n \in \omega\}$ is an ω-cover of X for a sequence $\{y_n : n \in \omega\} \subset Z$ of distinct elements. Thus there exists an increasing sequence n_k such that $Z \subset \bigcup_m \bigcap_{k>m} C_{n_k}$, so $X \subset \bigcup_m \bigcap_{k>m} C_{n_k}$.

Lemma 2. *Assume Martin's Axiom. Let $X \subset (0, 1)$ and $D, D_1 \subset (1, 2)$ be such that $|D| = |D_1| < \mathbf{c}$. Then $X \cup D$ is a γ-set iff $X \cup D_1$ is a γ-set.*

Proof. Assume that $X \cup D$ is a γ-set. Let $g : D \to D_1$ be a bijection and $f : X \cup D \to X \cup D_1$, $f(x) = x$ if $x \in X$ and $g(x)$ if $x \in D$. Let $J = \{O^n : n \in \omega\}$ be an open ω-cover of $X \cup D_1$. Let $O^n = O_1^n \cup O_2^n$ where $O_1^n \subset X$, $O_2^n \subset D_1$. Then $\{f^{-1}(O^n) : n \in \omega\}$ is an ω-cover of $X \cup D$, and we let $f^{-1}(O^n) = f^{-1}(O_1^n) \cup f^{-1}(O_2^n) = G_1^n \cup G_2^n$ where $G_1^n \subset X$ is open and $G_2^n \subset D$ G_δ-set. There exists a family of open subsets of D, $\{H_k^n : n, k \in \omega\}$, such that for every n, k $H_{k+1}^n \subset H_k^n$ and $G_2^n = \bigcap_k H_k^n$.

For $d \in D$ let $h_d : \omega \to \omega$ be such that for every n, if $d \notin G_2^n$, then $h_d(n) = m$ such that $d \notin H_m^n$ and if $d \in G_2^n$, then $h_d(n) = 0$. Since $|D| < \mathbf{c}$, there exists a function $h : \omega \to \omega$ such that for every $d \in D$, $h_d <^* h$. Observe that $\{G_1^n \cup H_{h(n)}^n : n \in \omega\}$ is an open ω-cover of $X \cup D$ so there exists an increasing sequence n_k such that $X \cup D = \bigcup_m \bigcap_{k>m} G_1^{n_k} \cup H_{h(n_k)}^{n_k}$. We will show that $X \cup D = \bigcup_m \bigcap_{k>m} G_1^{n_k} \cup G_2^{n_k}$. Evidently $X \subset \bigcup_m \bigcap_{k>m} G_1^{n_k} \cup G_2^{n_k}$. Suppose that there exists $d \in D$ such that $\forall_m \exists_{k \ge m} d \notin G_2^{n_k}$. But we know that $\exists_m \forall_{k \ge m} d \notin G_2^{n_k} \Rightarrow d \notin H_{h(n_k)}^{n_k}$, so $\forall_m \exists_{k \ge m} d \notin H_{h(n_k)}^{n_k}$, thus $d \notin \bigcup_m \bigcap_{k>m} H_{h(n_k)}^{n_k}$.

It follows that

$$X \cup D_1 = \bigcup_m \bigcap_{k>m} O^{n_k}.$$

Lemma 3. *Assume Martin's Axiom. If every subset of* $X \subset \mathbf{R}$ *is a* γ*-set, then for every* D *of cardinality less than continuum* D *is* F_σ *and* G_δ *in* $X \cup D$.

Proof. If $|X| < \mathbf{c}$, then the proof is well known. Assume that $|X| = \mathbf{c}$. Suppose that there exists D such that $|D| < \mathbf{c}$ and D is not G_δ or not F_σ in $X \cup D$. Then there exists an interval (a, b) such that $D \cap (a, b)$ is not G_δ or not F_σ in $(X \cup D) \cap (a, b)$ and $|X \backslash (a, b)| = \mathbf{c}$. There exists $D_1 \subset X \backslash (a, b)$ such that $|D_1| = |D \cap (a, b)|$. From Theorem 3 of Galvin and Miller [2] $[(X \cap (a, b)) \backslash D] \cup [(D \cap (a, b)) + (b - a)]$ is not a γ-set. Thus from Lemma 2 $((X \cap (a, b)) \backslash D) \cup D_1$ is also not a γ-set but $[((X \cap (a, b)) \backslash D) \cup D_1] \subset X$.

D in the Theorem is not Borel in X, so from Lemmas 1 and 3 we get the theorem.

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