PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 11, November 1995

LOCALLY COMPACT GROUPS WHICH HAVE THE WEAKLY COMPACT HOMOMORPHISM PROPERTY

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(Communicated by J. Marshall Ash)

ABSTRACT. A locally compact group G is WCHP if every weakly compact homomorphism from $L^1(G)$ into a Banach algebra has finite-dimensional range, and is WCHP^+ if every extension of G by an abelian group is WCHP. We verify the WCHP^+ property for certain locally compact groups, including all Moore groups and all connected groups.

A locally compact group G is said to have the *weakly compact homomorphism property*, or, shortly, to be WCHP, if every weakly compact homomorphism from $L^1(G)$ into a Banach algebra has finite-dimensional range ([Joh]). If every extension of G by an abelian group is WCHP, G is called WCHP^+ ([Joh, Definition 4.1]). All abelian and all compact groups are WCHP^+, and every extension of a WCHP^+ group by a WCHP^+ group is again WCHP^+ ([Joh, Corollary 4.5]). It is possible that every locally compact group is WCHP^+.

In this brief note, we use the heredity results for WCHP^+ groups from [Joh] and structure theorems for certain locally compact groups to establish the WCHP^+ property for these groups.

Theorem. *Let G be a locally compact group, and suppose that any of the following holds*:

(i) $G \in [FD]^- \cap [SIN]$.
(ii) $G \in [\mathrm{Moore}]$.
(iii) G *is almost connected.*

Then G is WCHP^+.

For the definitions of $[FD]^-$, $[SIN]$ and $[\mathrm{Moore}]$, see [Pal], for example.

Proof. Let $G \in [FD]^- \cap [SIN]$. By [G–M, Theorem 4.6], we have $G \cong V \times H$, where V is a vector group and H is an extension of a compact group by an abelian one. Hence, applying [Joh, Corollary 4.5] twice yields that G is WCHP^+.

Now, let $G \in [\mathrm{Moore}]$. Then G is a finite extension of a Takahashi group ([Rob]). Since Takahashi groups belong to $[FD]^- \cap [SIN]$, it follows from (i) and [Joh, Theorem 4.2(i)] that G is WCHP^+.

Received by the editors February 14, 1994 and, in revised form, April 13, 1994.
1991 *Mathematics Subject Classification.* Primary 43A20; Secondary 22D05, 46H99, 47B48.

Finally, let G be almost connected. Since G is the extension of a connected group by a compact one, it suffices in view of [Joh, Corollary 4.5] to assume that G is connected. By [M–Z, Theorem in Section 4.6], G is the extension of a connected Lie group by a compact group. Again using [Joh, Corollary 4.5], we may therefore assume that G is a connected Lie group. Now, by [Wil, Theorem 1.2], there are abelian subgroups $H_1, \ldots, H_n$ of G such that $G = H_1 \cdots H_n$. Hence, as a consequence of [Joh, Theorem 4.2(ii)], G is WCHP^+. □

Remarks. 1. In view of the remark in [Joh] preceding Corollary 4.5, our result shows that every connected group is even WCHP^{++}, a property formally stronger than WCHP^+.

2. In [Joh], B. E. Johnson asks:

> Is it possible [...] to show that if every irreducible representation of $L^1(G)$ is finite-dimensional, then G is WCHP?

If "irreducible representation" is supposed here to mean "topologically irreducible *-representation on a Hilbert space", as is usual in the group algebra context, this question asks if every $G \in$ [Moore] is WCHP and is thus resolved by our theorem.

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