PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 11, November 1995

A DIRECT PROOF OF A THEOREM OF TELGÁRSKY

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(Communicated by Andreas R. Blass)

ABSTRACT. We give a direct proof of the fact that if player TWO of a certain infinite game on a metric space has a winning strategy, then the space is a union of countably many of its compact subsets.

In an old paper [3] Menger introduced a property for topological spaces, which is nowadays known as *Menger's property.* In [2], Hurewicz reformulated Menger's property in a way that lends itself naturally to game-theoretic formulation: A space is said to have the Menger property if there is for every sequence $(\mathcal{G}_1, \dots, \mathcal{G}_n, \dots)$ of open covers of X, a sequence $(\mathcal{H}_1, \dots, \mathcal{H}_n, \dots)$ such that each $\mathcal{H}_n$ is a finite subset of $\mathcal{G}_n$ and $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ is an open cover of X. The Menger game is played as follows: Players ONE and TWO play an inning for each positive integer n. In the n-th inning, ONE chooses an open cover $\mathcal{U}_n$ of X and TWO responds by choosing a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. Player TWO wins the play $(\mathcal{U}_1, \mathcal{V}_1, \dots, \mathcal{U}_n, \mathcal{V}_n, \dots)$ if $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ is a cover for X; otherwise, ONE wins. We denote this game $M(X)$. The earliest to study this game was Hurewicz himself. In Theorem 10 of his 1925 paper [2], he proves (what amounts to): *For a set X of real numbers, ONE does not have a winning strategy in $M(X)$ if, and only if, X has the Menger property.*

Of course, Hurewicz did not use any game-theoretic terminology. On p. 171 of his paper [4] Telgársky explicitly formulated the Menger game. Telgársky called this game the Hurewicz game; seeing that the form of Menger's property which we are using is Hurewicz's reformulation of it, there is some merit to this choice of names. However, since this usage is not firmly established yet and since there is another game which is also called Hurewicz's game, which is directly related to a property introduced by Hurewicz, we think that it might be more useful for future reference to call the game discussed here the Menger game.

It is easy to see that if TWO has a winning strategy in the Menger game, then X has the Menger property. It is also easy to see that if a space is a union of countably many of its compact subsets, then TWO has a winning strategy in the Menger game. On p. 443 of [3], Menger actually conjectured that a space

Received by the editors March 18, 1994 and, in revised form, April 7, 1994.
1991 *Mathematics Subject Classification.* Primary 90D44; Secondary 04A99.
Research supported in part by Idaho State Board of Education grant 94-051.

would have the Menger property only if it is a union of countably many of its compact subsets. Fremlin and Miller disproved this conjecture by showing that there always are subsets of the real line which have Menger's property, but are not a union of countably many of their compact subsets ([1], Theorem 4). Interestingly, if we inquire which metric spaces are such that TWO has a winning strategy in the Menger game, we find that these are exactly the ones which are a union of countably many of their compact subsets ([4], Corollary 4). Telgársky's proof of this fact is fairly indirect. We give a more direct proof of Telgársky's result.

Recall that a topological space X is *H-closed* if, for every open cover $\mathscr{U}$ of X, there is a finite subset $\mathscr{V} \subseteq \mathscr{U}$ such that $X \subseteq \bigcup\{\overline{U} : U \in \mathscr{V}\}$. It is also well known that a regular topological space is compact if, and only if, it is H-closed. In particular, a metric space is compact if, and only if, it is H-closed.

Theorem 1. *Let X be a metric space such that TWO has a winning strategy in $M(X)$. Then X is a union of countably many of its compact subsets.*

Proof. Since TWO has a winning strategy in $M(X)$, X is a Lindelöf space. Since X is a metric space, it is second countable. Let $\mathscr{B}$ be a countable basis of open subsets of X for the topology of X. We let $\mathfrak{A}$ denote the collection of open covers of X which consist of countably many elements of $\mathscr{B}$. Let σ be a winning strategy for TWO.

For each $\tau \in {}^{<\omega}\omega$, define a subset C_τ of X and an element $\mathscr{G}_\tau$ of $\mathfrak{A}$ as follows:

(1) $C_\emptyset = \bigcap\{\overline{\bigcup \sigma(\mathscr{G})} : \mathscr{G} \in \mathfrak{A}\}$.

(2) Observe that $\{\overline{\bigcup \sigma(\mathscr{G})} : \mathscr{G} \in \mathfrak{A}\}$ is a countable collection; accordingly, choose elements $\mathscr{G}_{(n)}$, $n < \omega$, from $\mathfrak{A}$ such that $C_\emptyset = \bigcap\{\overline{\bigcup \sigma(\mathscr{G}_{(n)})} : n < \omega\}$.

(3) Assume that C_τ as well as $\mathscr{G}_{\tau\frown(n)}$, $n < \omega$, have been defined, and set

$$C_{\tau\frown(n)} = \bigcap\{\overline{\bigcup \sigma(\mathscr{G}_{(\tau(0))}, \dots, \mathscr{G}_\tau, \mathscr{G}_{\tau\frown(n)}, \mathscr{G})} : \mathscr{G} \in \mathfrak{A}\}.$$

(4) With $C_\tau = \bigcap\{\overline{\bigcup \sigma(\mathscr{G}_{(\tau(0))}, \dots, \mathscr{G}_\tau, \mathscr{G})} : \mathscr{G} \in \mathfrak{A}\}$, we see that there are countably many elements, say $\mathscr{G}_{\tau\frown(n)}$, $n < \omega$, of $\mathfrak{A}$ such that

$$C_\tau = \bigcap\{\overline{\bigcup \sigma(\mathscr{G}_{(\tau(0))}, \dots, \mathscr{G}_{\tau\frown(n)})} : n < \omega\},$$

which defines $\mathscr{G}_{\tau\frown(n)}$ for each n.

We now prove two things: (1) $X = \bigcup\{C_\tau : \tau \in {}^{<\omega}\omega\}$, and (2) each C_τ is a compact subset of X.

To see that (1) is true, suppose on the contrary that

$$x \in X \backslash (\bigcup\{C_\tau : \tau \in {}^{<\omega}\omega\}).$$

Then select $n_0, n_1, \dots, n_k, \dots$ such that $x \notin \overline{\bigcup \sigma(\mathscr{G}_{(n_0)}, \dots, \mathscr{G}_{(n_0, \dots, n_k)})}$ for each k. Then TWO loses the play

$$(\mathscr{G}_{(n_0)}, \sigma(\mathscr{G}_{(n_0)}), \dots, \mathscr{G}_{(n_0, \dots, n_k)}, \sigma(\mathscr{G}_{(n_0)}, \dots, \mathscr{G}_{(n_0, \dots, n_k)}), \dots),$$

despite having used the winning strategy σ—a contradiction.

To prove (2), fix a finite sequence τ of finite ordinals. First, observe that by its definition C_τ is closed. To see that C_τ is a compact subset of X, it suffices to show that it is H-closed. It also suffices to consider open covers of C_τ by elements of $\mathscr{B}$ only. Let $\mathscr{U}$ be such an open cover of C_τ. Let $\mathscr{V}$ be a cover of $X\backslash C_\tau$ by open subsets of $X\backslash C_\tau$ from $\mathscr{B}$, each having closure disjoint from C_τ (this is possible since X is regular). Then $\mathscr{G} = \mathscr{U} \cup \mathscr{V}$ is an element of $\mathfrak{A}$. Thus, $\sigma(\mathscr{G}_{(\tau(0))}, \ldots, \mathscr{G}_\tau, \mathscr{G})$ is a finite subset of $\mathscr{G}$ such that the union of the closure of the elements of this finite set covers C_τ. But the closures of elements of $\mathscr{V}$ belonging to this finite family are disjoint from C_τ. Discard the elements of $\mathscr{V}$ which belong to this finite family and denote the result $\mathscr{H}$. We see that the open cover $\mathscr{U}$ of C_τ has a finite subset $\mathscr{H}$ such that $C_\tau \subseteq \overline{\bigcup \mathscr{H}}$. This shows that C_τ is H-closed and completes the proof of the theorem. □

Regarding well-studied small sets of real numbers, Hurewicz's theorem and Telgársky's theorem give the following determinacy results for the Menger game.

Corollary 2. *If X is a Lusin set, then $M(X)$ is undetermined.*

Corollary 3. *If X is a Sierpiński set, then $M(X)$ is undetermined.*

As mentioned earlier, Theorem 4 of [1] implies that there is always a set of real numbers on which the Menger game is undetermined.

References

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