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POINTS JOINED BY THREE SHORTEST PATHS ON CONVEX SURFACES

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ABSTRACT. Let S be a convex surface and $x \in S$. It is shown here that the set of all points of S joined with x by at least three shortest paths can be dense in S. It is proven that, in fact, in the sense of Baire categories most convex surfaces have this property, for any x. Moreover, on most convex surfaces, for most of their points, there is just one farthest point (in the intrinsic metric), and precisely three shortest paths lead to that point.

1. INTRODUCTION

Let $S \subset \mathbb{R}^3$ be a (closed) convex surface and $x \in S$. The set T_x of all points joined with x by at least three *segments*, i.e., shortest paths in S, is known—and easily seen—to be at most countable. The set C_x of all points joined with x by at least two segments, is not very large either. It is proven in [6] that C_x is σ-porous and therefore of first Baire category and of (2-dimensional) Hausdorff measure 0. However, C_x must be uncountable if it contains more than one point, because it is arcwise connected [9].

The set C_x can be dense in S; in fact this is the generic behaviour, as shown in [6].

About T_x even the following question seems to be an open problem: "Does every convex surface S possess a point x with $T_x \neq \varnothing$?" This problem and a rapid investigation of common convex surfaces already show how thin T_x usually is. The first result of this paper is therefore very surprising, at least for the author. It states that T_x can be dense in S, too, and that this is the generic behaviour!

H. Steinhaus raised the more restrictive, exciting question whether there always exists a point $x \in S$ admitting a farthest point in T_x ([3], p. 44 (iii)). A related question of Steinhaus, less concrete, asks for a description of F_x, the set of all farthest points from $x \in S$ ([3], p. 44 (iv)). A few steps towards an answer were made in [9], where we proved, for example, that $F_x \subset \overline{C_x}$ and, if S is of class C^1, $F_x \subset C_x$. Of course, in general $F_x \not\subset T_x$. However, it may happen that $F_x \subset T_x$. We will show that this happens indeed, generically, for most points $x \in S$.

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We recall here that a property is said to be *generic* if it is shared by *most* elements of a Baire space, i.e., by all elements except those in a first category set.

The space $\mathcal{S}$ of all (closed) convex surfaces in $\mathbb{R}^3$ with the usual Pompeiu-Hausdorff distance and each surface $S \in \mathcal{S}$ with its intrinsic metric are examples of Baire spaces. The Pompeiu-Hausdorff distance will also be used between segments on convex surfaces.

For generic results in convexity see the surveys [4], [7].

2. Prerequisites

We shall make use of the following lemma.

Lemma 1. *Let $S \in \mathcal{S}$ and $x \in S$, suppose y, z are distinct points in C_x and consider two segments from x to y and another two from x to z. Then there is a Jordan arc A joining y and z, lying (except for the endpoints y and z) in the domain (i.e., connected open set) of S which has all four preceding segments on its boundary, and decomposing it into two subdomains Δ, Δ', such that each point of A is joined with x by a segment in $\overline{\Delta}$ and another segment in $\overline{\Delta'}$.*

This lemma follows immediately from Theorem 1 in [9] and its proof. We shall also need the following result, first proven in [9].

Lemma 2. *Let $S \in \mathcal{S}$, $x \in S$ and $y \in F_x$. Then any angle between two tangent directions at y measuring (on the tangent cone) more than π contains the tangent direction of a segment from y to x. Thus, if the full angle of S at y is larger than π, then $y \in C_x$.*

A basic result of Aleksandrov [1] on the convergence of angles will also be needed.

Lemma 3. *Let $S, S_n \in \mathcal{S}$, $x \in S$, $x_n \in S_n$, and assume that the full angle of S at x is 2π, x is an endpoint of the segments Σ, $\Sigma' \subset S$ and x_n is an endpoint of the segments Σ_n, $\Sigma'_n \subset S_n$. If S_n converges to S, x_n converges to x, Σ_n converges to Σ and Σ'_n converges to Σ', then the angle between Σ_n and Σ'_n at x_n converges to the angle between Σ and Σ' at x.*

For any Jordan arc J with definite directions at its endpoints the notions of a right and a left swerves can be introduced (see, for example, [2], pp. 108–110).

Let $M_1, \dots, M_n$ be two-dimensional manifolds, each with its own intrinsic metric. For every i consider an open set D_i with $\overline{D_i} \subset M_i$ and whose boundary is the union of pairwise disjoint rectifiable Jordan curves $C_1^i, \dots, C_{n_i}^i$. We say that the manifold M is obtained by *gluing* together $\overline{D_1}, \dots, \overline{D_n}$ if all C_l^i are decomposed into Jordan arcs which are pairwise identified in such a way that any two identified subarcs of these identified Jordan arcs have the same length, while $\overline{D_1} \cup \dots \cup \overline{D_n} = M$. We shall also make use of the following fundamental result of Aleksandrov [1].

Aleksandrov's gluing theorem. *Let $M_1, \dots, M_n$ have nonnegative curvature, and let the swerve have bounded variation on any subarc of any C_l^i. The manifold M obtained by gluing together $\overline{D_1}, \dots, \overline{D_n}$ has nonnegative curvature if and only if for any identified subarcs $A^i \subset C_l^i$ and $A^j \subset C_k^j$ the sum of the swerve of A^i in M_i towards D_i and the swerve of A^j in M_j towards D_j is nonnegative* *and for any point p belonging to more than two sets $\overline{D_i}$ the sum of the angles of these $\overline{D_i}$ at p is at most 2π.*

3. Generic density of T_x

The purpose of this section is to prove the following.

Theorem 1. *On most convex surfaces $S \in \mathscr{S}$, for each point $x \in S$, the set T_x is dense in S.*

Proof. Consider a surface $S \in \mathscr{S}$ satisfying $\overline{C_x} = S$ for some point $x \in S$. Let $O \subset S$ be open. We choose a point $y \in O \cap C_x$ and another point $y' \in C_x$. Let Σ_y, Σ'_y be two segments from x to y and $\Sigma_{y'}$, $\Sigma'_{y'}$ another two from x to y'. By Lemma 1, there is a Jordan arc $A \subset C_x$ joining y to y' with the following properties:

- (i) A lies, except for its endpoints, in the domain of S bounded by $\Sigma_y \cup \Sigma'_y \cup \Sigma_{y'} \cup \Sigma'_{y'}$ and decomposes it into two subdomains Δ_0 and Δ'_0.
- (ii) Each point of A is joined with x by (at least) two segments, which lie, except for their endpoints, one in Δ_0, the other in Δ'_0.

Suppose w.l.o.g. that $\Sigma_y \cup \Sigma_{y'} \cup A$ is the boundary of Δ_0 and $\Sigma'_y \cup \Sigma'_{y'} \cup A$ the boundary of Δ'_0. Since A is locally connected, there is a point $z \in A \cap O$ such that the whole subarc A' of A from y to z lies in O. Let

$$\Sigma_z \subset \overline{\Delta_0}, \qquad \Sigma'_z \subset \overline{\Delta'_0}$$

be two segments joining x with z. Denote by Δ the domain of S which is bounded by $\Sigma_y \cup \Sigma_z \cup A'$ and does not meet Σ'_y. Similarly, let Δ' be the domain bounded by $\Sigma'_y \cup \Sigma'_z \cup A'$ and not meeting Σ_y.

Choose a point $u \in \Delta \cap C_x$. Let τ_u, τ'_u be the tangent directions of two segments from x to u, at x. Also, let τ_y and τ_z be the tangent directions at x of Σ_y and Σ_z, respectively. We may suppose w.l.o.g. that τ_y, τ_u, τ'_u, τ_z lie in this order on the closed Jordan curve of all tangent directions at x.

Let now the point v move on A' from y to z, join it in $\overline{\Delta}$ by a segment Σ_v with x and consider the tangent direction τ_v of Σ_v at x. Since $\tau_v = \tau_y$ for $v = y$, $\tau_v = \tau_z$ for $v = z$ and τ_v never lies between τ_u and τ'_u, it follows that there is a a point $w \in A'$ which is a limit point of points v with τ_v between τ_y and τ_u and, also, of points v with τ_v between τ'_u and τ_z. Therefore w is joined by two distinct segments with x in $\overline{\Delta}$.

Since there is, in addition, a segment from w to x in $\overline{\Delta'}$, we have $w \in T_x$. This shows that $O \cap T_x \neq \varnothing$. Hence $\overline{T_x} = S$.

By Corollary 2 in [6], most surfaces $S \in \mathscr{S}$ satisfy $\overline{C_x} = S$ for any $x \in S$. This ends the proof.

4. Generic existence of points in $F_x \cap T_x$

We treat generically here Steinhaus' first question. This is less spectacular, and less easy too. Concretely, we obtain the following result.

Theorem 2. *On most surfaces $S \in \mathscr{S}$, for most points $x \in S$, the set F_x consists of a single point, joined with x by precisely three segments.*

Proof. Let $\mathbf{S}_x$ denote the set of all segments from x to points in F_x. Also, for any $S \in \mathscr{S}$, let

$$A_0(S) = \{x \in S\colon \operatorname{card} \mathbf{S}_x \leq 2\},$$
$$A_n(S) = \{x \in S\colon \text{there are four segments in } \mathbf{S}_x \text{ at mutual distances at least } n^{-1}\},$$
$$B_n(S) = \{x \in S\colon \operatorname{diam} F_x \geq n^{-1}\}.$$

For any number $n \in \mathbb{N}$, both sets A_n and B_n are closed in S. Let

$$\mathscr{S}' = \{S \in \mathscr{S}\colon \{x \in S\colon \operatorname{card} F_x \neq 1 \text{ or } \operatorname{card} \mathbf{S}_x \neq 3 \text{ is of 2nd category}\},$$
$$\mathscr{A}_n = \{S \in \mathscr{S}\colon A_n(S) \text{ is not nowhere dense}\} \qquad (n \in \{0\} \cup \mathbb{N}),$$
$$\mathscr{B}_n = \{S \in \mathscr{S}\colon B_n(S) \text{ is not nowhere dense}\} \qquad (n \in \mathbb{N}).$$

We show that $\mathscr{S}'$ is of first category. We observe, indeed, that

$$\mathscr{S}' \subset \bigcup_{n=0}^{\infty} \mathscr{A}_n \cup \bigcup_{n=1}^{\infty} \mathscr{B}_n .$$

We shall say that a point in $\mathbb{R}^3$ is *rational* if its coordinates are rational. Also, for $S \in \mathscr{S}$ and $z \in \mathbb{R}^3$, let $\delta(z, S) = \min_{x \in S} \|x - z\|$. If a closed subset of S is not nowhere dense, it must include a disc D (in the intrinsic metric of S) on S. For every such disc D we may find a rational point $z \in \mathbb{R}^3$, at distance at most q^{-1} from S $(q \in \mathbb{N})$, such that $B(z, 2q^{-1}) \cap S \subset D$. So, for each $m \in \mathbb{N}$,

$$\mathscr{A}_0 \cup \mathscr{A}_m \cup \mathscr{B}_m \subset \bigcup_{z,q} \mathscr{A}_{m,z,q},$$

where, for any $m, q \in \mathbb{N}$ and rational $z \in \mathbb{R}^3$,

$$\mathscr{A}_{m,z,q} = \{S \in \mathscr{A}_0 \cup \mathscr{A}_m\colon \delta(z, S) \leq q^{-1} \text{ and } B(z, 2q^{-1}) \cap S \text{ is included in } \overline{A_0} \text{ or } A_m \text{ or } B_m\}.$$

To show that $\mathscr{A}_{m,z,q}$ is nowhere dense, let $\mathscr{O} \subset \mathscr{S}'$ be open. If $\mathscr{A}_{m,z,q} \cap \mathscr{O} = \varnothing$, there is nothing to show. If $S_0 \in \mathscr{A}_{m,z,q} \cap \mathscr{O}$, we choose a polytopal surface in $\mathscr{O}$ approximating S_0, with a vertex x in $S_0 \cap B(z, 2q^{-1})$ and with small spherical images at all vertices. Then, by Lemma 2, every arc of length π on the Jordan curve J (of length close to 2π) of all tangent directions at some point $y \in F_x$ contains the tangent direction of a segment in $\mathbf{S}_x$. It follows that there are two segments from x to y such that the lengths of the arcs determined by their tangent directions at y on J are at most π, or there are three segments from x to y such that the lengths of the three arcs in which their tangent directions at y divide J are less than π.

In the case of two segments, we consider a small number $\alpha > 0$ and a small triangle $\boldsymbol{\Theta}$ of angles α, α, $\pi - 2\alpha$ and sides a, a, b, say. We also consider two congruent isosceles triangles with sides r, r, a, where r equals the distance from x to y on S, and a third isosceles triangle with sides r, r, b and a small angle 3α. Now cut P along the two segments from x to y. By Aleksandrov's gluing theorem, for α sufficiently small, the four triangles and the two pieces of P can be glued in an obvious way together, the two congruent long thin isosceles triangles becoming adjacent. The nonnegative swerve of a segment on both sides and our choice of the angles and sides of the involved triangles imply the assumptions in Aleksandrov's gluing theorem. This construction works even if y lies on a facet of P. This cannot happen in the present case of two segments if $\operatorname{card} \mathbf{S}_x \leq 3$ but may well happen if $\operatorname{card} \mathbf{S}_x \geq 4$, for instance if there are precisely four segments from x to y and the tangent directions at y are pairwise opposite.

In the case of three segments, we consider the three distances α, β, γ determined on J by their tangent directions. We build a small triangle $\boldsymbol{\Theta}$ with sides a, b, c and angles $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$ and three isosceles triangles with sides r, r, a; r, r, b; r, r, c. We cut P along the three segments and glue the resulting three pieces together with the four preceding triangles such that the piece containing the angle α is glued to the isosceles triangles of smallest sides b and c, etc, $\boldsymbol{\Theta}$ being glued to all three isosceles triangles.

In both cases, if the initial triangle $\boldsymbol{\Theta}$ is chosen small enough, then the new polytopal surface P' can be chosen in $\mathscr{O}$, due to the Olovianishnikov-Pogorelov uniqueness theorem [5] (see the comment in [8], p. 114). Of course, the point of P' corresponding to x can be kept at x. For P', the set F_x consists of a single point y' inside the set corresponding (through the isometry) to $\boldsymbol{\Theta}$, at a distance from x larger than r (the choice of an angle of 3α for an isosceles triangle in the first case was made to this end). Clearly, in P', $\operatorname{card} \mathbf{S}_x = 3$, the full angle at y' is 2π and the angles between the directions at y' of the three segments are all less than π.

Consider a sequence of convex surfaces $\{S_n\}_{n=1}^{\infty}$ converging to P', $x_n \in S_n$, $y_n \in F_{x_n}$ and $x_n \to x$. Then, clearly, $y_n \to y'$ and any segment from x_n to y_n converges to a segment from x to y'. By Lemma 3, if Σ_n, Σ_n' are two such segments converging to Σ, Σ' respectively, then the angle between Σ_n and Σ_n' at y_n converges to the angle between Σ and Σ' at y'.

Suppose now $S_n \in \mathscr{A}_{m,z,q}$. Then $B(z, 2q^{-1}) \cap S_n$ is contained in at least one of the sets $\overline{A_0}$, A_m, B_m. This means that (choosing $x_n \in A_0$ in the first case) there are precisely two segments from x_n to y_n (whose angle at y_n must converge by Lemma 2 to π, because the full angle at y_n converges to 2π), or there are four segments in $\mathbf{S}_{x_n}$ at distance at least m^{-1} from each other, or the diameter of F_{x_n} is at least m^{-1}. But this implies that the angle between two of the three segments in $\mathbf{S}_x$ is π (in the first case), or there are four distinct segments in $\mathbf{S}_x$ (in the second case), or F_x has more than one point (in the third case), and contradictions are obtained.

Hence there is a ball around P' in $\mathscr{S}$ disjoint from $\mathscr{A}_{m,z,q}$. Thus $\mathscr{A}_{m,z,q}$ is nowhere dense, and $\mathscr{A}_0 \cup \mathscr{A}_m \cup \mathscr{B}_m$ of first category in $\mathscr{S}$. Hence $\mathscr{S}'$ is of first category, and the proof finds its end.

5. Open problems

We conclude the paper with open questions arising from the generic investigation of $\mathscr{S}$ and related to the results of this paper.

Problem 1. Is it true, for most convex surfaces $S \in \mathscr{S}$, that for any $x \in S$ and $y \in F_x$, x and y are joined by at most 3 segments?

Problem 2. Is it true, for most convex surfaces $S \in \mathscr{S}$, that any two points of S are joined by at most 3 segments?

Problem 3. Is it true, for most convex surfaces $S \in \mathscr{S}$, that (the) two farthest points of S are joined by precisely 3 segments?

Examples of polytopal surfaces show that two farthest points may be joined by 5 segments. It follows from recent unpublished work of P. Horja about manifolds of nonpositive curvature that this must be the case if the two points are not vertices.

Problem 4. Is the family of all convex surfaces with two farthest points joined by 5 segments dense in $\mathscr{S}$? Is 5 the right number in Problem 3?

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