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A FULL EXTENSION OF THE ROGERS-RAMANUJAN CONTINUED FRACTION

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ABSTRACT. In this paper, we present the natural extension of the Rogers-Ramanujan continued fraction to the nonterminating very well-poised basic hypergeometric function ${}_8\phi_7$. In a letter to Hardy, Ramanujan indicated that he possessed a four variable generalization. Our generalization has seven variables and is, perhaps, all one can expect from this method.

1. INTRODUCTION

One of the most intriguing results from classical q-series is the Rogers-Ramanujan continued fraction [1, p. 440]:

$$\prod_{n=0}^{\infty}\frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} = 1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}, \tag{1.1}$$

where $|q|<1$ throughout this paper.

The proof of (1.1) relies fundamentally on the following. Let

$$G(z,q)=G(z)=1+\sum_{n=1}^{\infty}\frac{z^n q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{1.2}$$

Then

(1.3)

$$G(z)=\prod_{n=1}^{\infty}(1-zq^n)^{-1} \times\left(1+\sum_{n=1}^{\infty}\frac{(1-zq)(1-zq^2)\cdots(1-zq^{n-1})(-1)^n z^{2n} q^{n(5n-1)/2}(1-zq^{2n})}{(1-q)(1-q^2)\cdots(1-q^n)}\right),$$

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and

$$\frac{G(z)}{G(zq)} = 1 + \cfrac{zq}{1 + \cfrac{zq^2}{1 + \cfrac{zq^3}{1 + \ddots}}}. \tag{1.4}$$

G. N. Watson [7] gave the following five-parameter generalization of (1.3) which is perhaps the most natural nonmultiple series generalization:

$$\begin{aligned} &{}_8\phi_7\left(\begin{matrix} z, q\sqrt{z}, -q\sqrt{z}, a_1, a_2, a_3, a_4, a_5; q, \frac{z^2q^2}{a_1a_2a_3a_4a_5} \\ \sqrt{z}, -\sqrt{z}, \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3}, \frac{zq}{a_4}, \frac{zq}{a_5} \end{matrix}\right) \\ &= \frac{(zq)_N(\frac{zq}{a_3a_4})_N}{(\frac{zq}{a_3})_N(\frac{zq}{a_4})_N}\, {}_4\phi_3\left(\begin{matrix} \frac{zq}{a_1a_2}, a_3, a_4, a_5; q, q \\ \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{a_3a_4a_5}{z} \end{matrix}\right) \end{aligned} \tag{1.5}$$

where $a_5 = q^{-N}$ with N a nonnegative integer, and

$$(a; q)_N = (a)_N = (1-a)(1-aq)\cdots(1-aq^{N-1}), \tag{1.6}$$

$${}_{r+1}\phi_r\left(\begin{matrix} a_0, a_1, \cdots, a_r; q, t \\ b_1, \cdots, b_r \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_0)_n(a_1)_n\cdots(a_r)_n t^n}{(q)_n(b_1)_n\cdots(b_r)_n}. \tag{1.7}$$

We remark that the expressions in (1.5) are most naturally viewed as functions of $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5}$ and are, in fact, continuous in each $\frac{1}{a_i}$ around zero, i.e. $a_i = \infty$. Consequently when we set any of $a_1, \dots, a_5 = \infty$ throughout this paper we shall be doing so in light of the above comments. We could, of course, have begun with each a_i replaced by its reciprocal; however this would not be consistent with standard notation [5, pp. 4, 35].

When $a_1 = a_2 = a_3 = a_4 = a_5 = \infty$ (i.e. $a_5 = \infty$ means $\lim_{N\to\infty} q^{-N}$), (1.5) reduces term-by-term to (1.3). In [1, p. 434], it is noted that the case $a_3 = a_4 = a_5 = \infty$ of (1.5) is quite possibly the function used in a general continued fraction identity alluded to by Ramanujan [6, p. xxviii]. One is naturally led to ask: Can one extend (1.4) wherein $G(z)$ is replaced by the very well-poised ${}_8\phi_7$ of (1.5)?

We answer this question affirmatively in Theorem 1. It is to be emphasized that Theorem 1 does *not* require any of $a_1, a_2, \dots, a_5$ to be of the form q^{-N}. Consequently our result holds for the nonterminating very well-poised ${}_8\phi_7$ first considered by W. N. Bailey in [4] (cf. [5, p. 42, eq. (2.10.10)]).

2. q-HYPERGEOMETRIC BACKGROUND

Our work is based on the q-difference equations for very well-poised basic hypergeometric series studied in [1] (cf. [2]). Thus we require many of the auxiliary functions defined there.

$$\begin{aligned} &C_{k,i}(a_1, a_2, \dots, a_\lambda; x; q) = C_{k,i}((a); x; q)_\lambda \\ &= \sum_{n\geq 0} (-1)^{n(\lambda+1)} x^{kn} (a_1a_2\cdots a_\lambda)^{-n} q^{((2k-\lambda+1)n^2+(\lambda+1)n-2in)/2} \\ &\qquad \frac{(1-x^iq^{2ni})}{(1-x)} \frac{(x)_n(a_1)_n(a_2)_n\cdots(a_\lambda)_n}{(q)_n(\frac{xq}{a_1})_n(\frac{xq}{a_2})_n\cdots(\frac{xq}{a_\lambda})_n}; \end{aligned} \tag{2.1}$$

$$H_{k,i}(a_1, a_2, \ldots, a_\lambda; x, q) = H_{k,i}((a); x; q)_\lambda = \frac{(\frac{xq}{a_1})_\infty(\frac{xq}{a_2})_\infty \cdots (\frac{xq}{a_\lambda})_\infty}{(xq)_\infty} C_{k,i}((a); x; q)_\lambda. \tag{2.2}$$

We note that [1, p. 434]

$$C_{k,1}((a); x; q)_{2k+1} = {}_{2k+4}\phi_{2k+3}\left(\begin{matrix} x, q\sqrt{x}, -q\sqrt{x}, a_1, a_2, \cdots, a_{2k+1}; q, \frac{x^k q^k}{a_1 a_2 \cdots a_{2k+1}} \\ \sqrt{x}, -\sqrt{x}, \frac{xq}{a_1}, \frac{xq}{a_2}, \cdots, \frac{xq}{a_{2k+1}} \end{matrix}\right). \tag{2.3}$$

The crucial relations among these functions are the following [1, p. 435, Theorem 1]:

$$H_{k,i}((a); x; q)_\lambda - H_{k,i-1}((a); x; q)_\lambda = x^{i-1}\sum_{j=0}^{\lambda}(-1)^j \sigma_j\left(\frac{1}{a_1}, \cdots, \frac{1}{a_\lambda}\right) x^j q^j H_{k,k+1-i-j}((a); xq; q)_\lambda, \tag{2.4}$$

where $\sigma_j(y_1, \ldots, y_\lambda)$ is the j-th elementary symmetric function of $y_1, \ldots, y_\lambda$;

$$H_{k,-i}((a); x; q)_\lambda = -x^{-i} H_{k,i}((a); x; q)_\lambda; \tag{2.5}$$

$$H_{k,0}((a); x; q)_\lambda \equiv 0. \tag{2.6}$$

Henceforward we are only interested in the case $k = 2$, $\lambda = 5$. Given these values of k and λ, we abbreviate

$$H_i(x) = H_{2,i}(a_1, a_2, a_3, a_4, a_5; x; q) \tag{2.7}$$

and

$$\sigma_j = \sigma_j\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5}\right). \tag{2.8}$$

The four instances of (2.4) with $i = 2, 1, 0, -1$ reduce to the following once we use (2.5) and (2.6) for simplification:

$$H_2(x) - H_1(x) - \sigma_3 x^2 q H_2(xq) + x^2 q \sigma_4 H_3(xq) - x^2 q \sigma_5 H_4(xq) = (x - x^2 q \sigma_2) H_1(xq); \tag{2.9}$$

$$H_1(x) - (1 - x^2 q^2 \sigma_4) H_2(xq) - x^2 q^2 \sigma_5 H_3(xq) = (-xq\sigma_1 + x^2 q^2 \sigma_3) H_1(xq); \tag{2.10}$$

$$H_1(x) + (xq\sigma_1 - x^3 q^3 \sigma_5) H_2(xq) - H_3(xq) = (x^2 q^2 \sigma_2 - x^3 q^3 \sigma_4) H_1(xq); \tag{2.11}$$

$$H_2(x) - xH_1(x) - x^2 q^2 \sigma_2 H_2(xq) + xq\sigma_1 H_3(xq) - H_4(xq) = (-x^3 q^3 \sigma_3 + x^4 q^4 \sigma_5) H_1(xq). \tag{2.12}$$

These four functional equations will be used in Section 3 to obtain the relevant second order q-difference equation for $H_{2,1}(a_1, a_2, a_3, a_4, a_5; x; q)$.

3. The main results

Our next step consists of replacing x by xq in each of (2.9)–(2.12). We rewrite the resulting equations so that $H_1(xq)$ and $H_1(xq^2)$ appear on the right of each equation.

$$H_2(xq) - \sigma_3 x^2 q^3 H_2(xq^2) + x^2 q^3 \sigma_4 H_3(xq^2) - x^2 q^3 \sigma_5 H_4(xq^2) = H_1(xq) + (xq - x^2 q^3 \sigma_2) H_1(xq^2); \tag{3.1}$$

$$-(1 - x^2 q^4 \sigma_4) H_2(xq^2) - x^2 q^4 \sigma_5 H_3(xq^2) = (-xq^2\sigma_1 + x^2 q^4 \sigma_3) H_1(xq^2) - H_1(xq); \tag{3.2}$$

$$(xq^2\sigma_1 - x^3 q^6 \sigma_5) H_2(xq^2) - H_3(xq^2) = (x^2 q^4 \sigma_2 - x^3 q^6 \sigma_4) H_1(xq^2) - H_1(xq); \tag{3.3}$$

$$H_2(xq) - x^2 q^4 \sigma_2 H_2(xq^2) + xq^2 H_3(xq^2) - H_4(xq^2) = (-x^3 q^6 \sigma_3 + x^4 q^8 \sigma_5) H_1(xq^2). \tag{3.4}$$

Equations (2.9)–(2.12), (3.1)–(3.4) constitute a system of eight linear equations in the eight unknowns: $H_1(x)$, $H_2(x)$, $H_2(xq)$, $H_2(xq^2)$, $H_3(xq)$, $H_3(xq^2)$, $H_4(xq)$, $H_4(xq^2)$. The system turns out to be nonsingular, and consequently Cramer's Rule assisted by MACSYMA allows us to solve for $H_1(x)$ which upon inspection reveals a linear relation among $H_1(x)$, $H_1(xq)$ and $H_1(xq^2)$. To make this relation most succinct we introduce

$$p(x) = p(x; \sigma_1, \sigma_4, \sigma_5, q) = 1 - x^2 q^2 \sigma_4 + x^3 q^3 \sigma_1 \sigma_5 - x^5 q^5 \sigma_5^2. \tag{3.5}$$

Using this notation, we find that after simplification and division by $(-1 + x^2 q \sigma_5)$

$$Q(x) H_1(x) = P(x) H_1(xq) + R(x) H_1(xq^2), \tag{3.6}$$

where

$$Q(x) = (1 - x^2 q^2 \sigma_5)(1 - x^2 q^3 \sigma_5) p(xq), \tag{3.7}$$

(3.8)
$$\begin{aligned} P(x) = &-xq(1 - x^2 q^3 \sigma_5) p(xq)(\sigma_1 - xq\sigma_3 + x^3 q^3 \sigma_2 \sigma_5 - x^4 q^4 \sigma_4 \sigma_5) \\ &- p(x) \Big\{ (-1 - xq\sigma_1 + x^3 q^4 \sigma_5) p(xq) \\ &\quad - x^6 q^{11} \sigma_2 \sigma_5 + x^5 q^9 \sigma_1 \sigma_4 \sigma_5 - x^5 q^9 \sigma_4 \sigma_5 + x^4 q^7 \sigma_3 \sigma_5 \\ &\quad + x^4 q^7 \sigma_2 \sigma_5 - x^4 q^7 \sigma_4^2 - x^3 q^5 \sigma_1 \sigma_5 + x^2 q^3 \sigma_4 - x^2 q^3 \sigma_3 + xq\sigma_1 \Big\}, \end{aligned}$$

$$R(x) = xq\, p(x) \prod_{1 \leqq i < j \leqq 5} \left(1 - \frac{xq^2}{a_i a_j}\right). \tag{3.9}$$

Theorem 1.

$$\frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \cfrac{R(x)/Q(x)}{\cfrac{P(xq)}{Q(xq)} + \cfrac{R(xq)/Q(xq)}{\cfrac{P(xq^2)}{Q(xq^2)} + \cfrac{R(xq^2)/Q(xq^2)}{\cfrac{P(xq^3)}{Q(xq^3)} + \ddots}}}. \tag{3.10}$$

Proof. This assertion is merely the iteration of a restatement of (3.6) written in the form

$$\frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \frac{R(x)/Q(x)}{\frac{H_1(xq)}{H_1(xq^2)}}. \tag{3.11}$$

Convergence is guaranteed by the fact that $P(x)$, $Q(x)$ and $R(x)$ are polynomials in x, $Q(0)=1$, and $R(x)$ has no constant term. □

Related contiguous relations and continued fractions may be derived from (3.6). For example, if we put $a_5 = q^{-N}$ and let $N \to \infty$ in (1.5) we find

$$\begin{aligned} &H_1(a_1, a_2, a_3, a_4; x; q) \\ &= \left(\frac{xq}{a_1}\right)_\infty \left(\frac{xq}{a_2}\right)_\infty \left(\frac{xq}{a_3a_4}\right)_\infty {}_3\phi_2\left(\begin{matrix} \frac{xq}{a_1a_2}, & a_3, a_4 \\ \frac{xq}{a_1}, & \frac{xq}{a_2} \end{matrix} ; \frac{xq}{a_3a_4}\right). \end{aligned} \tag{3.12}$$

Making the change of variables

$$x \mapsto \frac{de}{aq}, \; a_1 \mapsto \frac{e}{a}, \; a_2 \mapsto \frac{d}{a}, \; a_3 \mapsto b, \; a_4 \mapsto c,$$

cancelling infinite products and simplifying the resulting polynomials give the following contiguous relation for a ${}_3\phi_2$:

$$\begin{aligned} S\,{}_3\phi_2\left(\begin{matrix} a & b & c \\ d & e & \end{matrix} ; \frac{de}{abc}\right) &= T\,{}_3\phi_2\left(\begin{matrix} aq, & b, & c \\ dq, & eq & \end{matrix} ; \frac{de}{abc}q\right) \\ &\quad + U\,{}_3\phi_2\left(\begin{matrix} aq^2, & b, & c \\ dq^2, & eq^2 & \end{matrix} ; \frac{de}{abc}q^2\right), \end{aligned} \tag{3.13}$$

where the polynomials S, T and U are

$$\begin{aligned} S &= bc(d)_2(e)_2(abc-de)(bc-deq^2), \\ T &= (1-dq)(1-eq)(bc-deq)[bcde(a(b+c)+d+e)(1+q) - ((b+c)de \\ &\qquad + abc(d+e))(bc+deq) + a(bc-de)(bc-deq^2)], \\ U &= de(bc-de)(1-aq)(b-dq)(c-dq)(b-eq)(c-eq). \end{aligned}$$

Applying the transformation (3.2.7) of [5] term-by-term to the contiguous relation and simplifying the products yield

$$\begin{aligned} S'\,{}_3\phi_2\left(\begin{matrix} a & b & c \\ d & e & \end{matrix} ; \frac{de}{abc}\right) &= T'\,{}_3\phi_2\left(\begin{matrix} aq, bq, cq \\ dq, \quad eq^2 \end{matrix} ; \frac{de}{abc}\right) \\ &= U'\,{}_3\phi_2\left(\begin{matrix} aq^2, & bq^2, & cq^2 \\ dq^2, & eq^4 & \end{matrix} ; \frac{de}{abc}\right), \end{aligned} \tag{3.14}$$

where now

$$\begin{aligned} S' &= a^2b^2c^2(d)_2(e)_2(1-eq^2), \\ T' &= abc(1-dq)(eq)_3(de(ab+ac+bc+e)(1+q) \\ &\qquad\qquad - d(abc+(a+b+c)e)(1+eq) \\ &\qquad\qquad\qquad + abc(1-e)(1-eq^2)), \\ U' &= d^2(1-e)e(1-aq)(1-bq)(1-cq)(a-eq)(b-eq)(c-eq). \end{aligned}$$

Iterating these two contiguous relations gives

$$\frac{{}_3\phi_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix}; \frac{de}{abc}\right)}{{}_3\phi_2\left(\begin{matrix} aq,\ b,\ c \\ dq,\ eq \end{matrix}; \frac{de}{abc} q\right)} = \frac{T(a,d,e)}{S(a,d,e)} + \frac{U(a,d,e)/S(a,d,e)}{T(aq,dq,eq)/S(aq,dq,eq)+\cdots} \tag{3.15}$$

and

$$\begin{aligned}\frac{{}_3\phi_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix}; \frac{de}{abc}\right)}{{}_3\phi_2\left(\begin{matrix} aq,\ bq,\ cq \\ dq,\ eq^2 \end{matrix}; \frac{de}{abc}\right)} &= \frac{T'(a,b,c,d,e)}{S'(a,b,c,d,e)} \\ &+ \frac{U'(a,b,c,d,e)/S'(a,b,c,d,e)}{T'(aq,bq,cq,dq,eq^2)/S'(aq,bq,cq,dq,eq^2)+\cdots}.\end{aligned} \tag{3.16}$$

For both continued fractions convergence follows from the fact that after cancelling common factors of powers of q from the partial numerators and denominators, the partial numerators tend to zero, while the partial denominators do not.

Obviously, although Theorem 1 is explicit, it nonetheless lacks the elegance of (1.1) or (1.4). If we let a_4 and a_5 tend to infinity (so that $\sigma_4 = \sigma_5 = 0$) and denote the resulting $P(x)$, $Q(x)$ and $R(x)$ by $\bar{P}(x)$, $\bar{Q}(x)$ and $\bar{R}(x)$ respectively, then we see that

$$\bar{Q}(x) = 1, \tag{3.17}$$

$$\bar{R}(x) = xq\left(1 - \frac{xq^2}{a_1a_2}\right)\left(1 - \frac{xq^2}{a_1a_3}\right)\left(1 - \frac{xq^2}{a_2a_3}\right), \tag{3.18}$$

$$\bar{P}(x) = 1 - \frac{xq}{a_1} - \frac{xq}{a_2} - \frac{xq}{a_3} + \frac{x^2q^2}{a_1a_2a_3}(1+q). \tag{3.19}$$

Thus we obtain

Corollary 1.

$$\begin{aligned}&\frac{H_{2,1}(a_1, a_2, a_3; x; q)}{H_{2,1}(a_1, a_2, a_3; q; q)} \\ &= \bar{P}(x) + \cfrac{\bar{R}(x)}{\bar{P}(xq) + \cfrac{\bar{R}(xq)}{\bar{P}(xq^2) + \ddots}}.\end{aligned} \tag{3.20}$$

Several remarks are now in order. First it is clear from (3.17)–(3.20) that (1.4) is the case $a_1 = a_2 = a_3 = a_4 = a_5 = +\infty$. Also the form for the $H_{k,i}$ given in (2.2) will yield the right-hand side of (1.3) and not the right-hand side of (1.2). However, for completeness, we note we may deduce from (2.2), (2.3) and (1.5) that

$$H_{2,1}(a_1, a_2, a_3; x; q) = \left(\frac{xq}{a_1a_3}\right)_\infty \left(\frac{xq}{a_2}\right)_\infty \sum_{n=0}^{\infty} \frac{(a_1)_n (a_3)_n \left(\frac{xq}{a_1a_3}\right)^n}{(q)_n \left(\frac{xq}{a_2}\right)_n}, \tag{3.21}$$

and this last expression reduces to the right-hand side of (1.2) as a_1, a_2 and $a_3 \to \infty$.

To conclude we examine the simplest case involving three finite parameters. We replace q by q^4 and then set $a_1 = q$, $a_2 = q^2$, $a_3 = q^3$ in Corollary 1. By (3.21) we see that

$$\begin{aligned}\lim_{x\to 1^-} & H_{2,1}(q, q^2, q^3; x; q^4) \\ &= \lim_{x\to 1^-} (x; q^4)_\infty (xq^2; q^4)_\infty \sum_{n=0}^{\infty} \frac{(q, q^2)_{2n} x^n}{(q^4; q^4)_n (xq^2; q^4)_n} = (q; q^2)_\infty,\end{aligned}$$

and if we define

$$f(q) = H_{2,1}(q, q^2, q^3; q^4, q^4),$$

then by (3.21) after simplification

$$\begin{aligned} f(q) &= \frac{(q^2; q^4)_\infty^2}{q(1-q)} \sum_{n=0}^{\infty} \frac{(q; q^2)_{2n+1} q^{2n+1}}{(q^2; q^2)_{2n+1}} \\ &= \frac{(q^2; q^4)_\infty^2}{q(1-q)} \sum_{n=0}^{\infty} \frac{(q; q^2)_n q^n}{(q^2; q^2)_n} \frac{(1-(-1)^n)}{2} \\ &= \frac{(q^2; q^4)_\infty^2}{2q(1-q)} \left(\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} - \frac{(-q^2; q^2)_\infty}{(-q; q^2)_\infty} \right) \\ &= \frac{(q^2; q^4)_\infty}{2q(1-q)} ((-q; -q)_\infty - (q; -q)_\infty). \end{aligned}$$

Hence by [3, p. 23, eq. (2.2.12)]

$$\begin{aligned} f(-q) &= \frac{-(q^2; q^4)_\infty}{2q(1+q)} (-q)_\infty \, 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \\ &= (-q^3; q^2)_\infty \sum_{n=1}^{\infty} (-1)^{n-1} q^{n^2-1}, \end{aligned}$$

and so

$$H_{2,1}(q, q^2, q^3; q^4; q^4) = (q^3; q^2)_\infty \sum_{n=0}^{\infty} q^{n^2+2n}.$$

Consequently Corollary 1 reduces to

(3.22)

$$\frac{1-q}{\sum_{n=0}^{\infty} q^{n^2+2n}} = 1 - q - q^3 + q^6 + \cfrac{q^4(1-q^3)(1-q^4)(1-q^5)}{1 - q^5 - q^6 - q^7 + q^{10} + q^{14} + \cfrac{q^8(1-q^7)(1-q^8)(1-q^9)}{1-q^9-q^{10}-q^{11}+q^{18}+q^{22}+ \ddots}}.$$

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