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VIEW-OBSTRUCTION PROBLEMS AND KRONECKER'S THEOREM

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ABSTRACT. In this paper we show how the quantitative forms of Kronecker's theorem in Diophantine approximations can be applied to investigate view-obstruction problems. In particular we answer a question in [Yong-Gao Chen, *On a conjecture in Diophantine approximation*, III, J. Number Theory **39** (1991), 91–103].

1. INTRODUCTION

In [7] Cusick proposed the view-obstruction problems for n-dimensional geometry. Here we do not give the original definitions. Instead, we give equivalent definitions. For details one may refer to Cusick [8] and the author [5]. The view-obstruction problem for cubes in E^n is to determine the value

$$\lambda(n) = 2 \sup_{a_1, \dots, a_n} \inf_{t\in[0,1]} \max_{1\le i\le n} \left\| a_i t - \frac{1}{2} \right\|, \tag{1}$$

where the supremum is taken over all n-tuples of positive integers $a_1, a_2, \dots, a_n$, and $\|x\|$ denotes the distance from x to the nearest integer. The view-obstruction problem for spheres in E^n is to determine the value

$$\nu(n) = 2 \left(\sup_{a_1, \dots, a_n} \inf_{t\in[0,1]} \sum_{i=1}^{n} \left\| a_i t - \frac{1}{2} \right\|^2 \right)^{1/2}, \tag{2}$$

where the supremum has the above meaning. We can derive that

$$\lambda(n) = 1 - 2k(n),$$

where

$$k(n) = \inf_{a_1, \dots, a_n} \sup_{t\in[0,1]} \min_{1\le i\le n} \|a_i t\|.$$

In [7] Cusick conjectured that $k(n) = (n+1)^{-1}$, and the corresponding value for $\lambda(n)$ is $(n-1)/(n+1)$. Up to now, we know that $\lambda(2) = 1/3$, $\lambda(3) = 1/2$, $\lambda(4) = 3/5$ (Cusick [7, 8]), $\nu(2) = 1/\sqrt{5}$ (Cusick [7]), $\nu(3) = \sqrt{3/7}$ (Dumir

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and Hans-Gill [9], also see author [4]), $\nu(4) = \sqrt{11/15}$ (author [5]). In [6] I considered

$$K_f(n) = \sup_{a_1, \dots, a_n} \inf_{t \in \mathbf{R}} \sum_{i=1}^{n} f(\|a_i t\|),$$

for non-negative real-valued functions $f(x)$ in $[0, 1]$, where the supremum is taken over all n-tuples of positive real numbers $a_1, a_2, \dots, a_n$. There we derived some properties of $K_f(n)$ and proved that

$$\lim_{n\to\infty} \frac{\nu(n)^2}{n} = \sup_n \frac{\nu(n)^2}{n},$$

$$\nu(n)^2 \le \frac{n}{3} - \frac{1}{45},$$

$$k(n) \ge (2n - 1 - (2n-3)^{-1})^{-1}, \qquad n \ge 5.$$

In [4] the following lemma is given.

Lemma 1. *Let $0 < \lambda \le \frac{1}{2}$ be a real number and $a_1, a_2, \dots, a_n$ be n positive real numbers. The following statements are equivalent:*

(A) *There exist n integers $k_1, k_2, \dots, k_n$ such that*

$$a_i k_j - a_j k_i \le (1-\lambda) a_j - \lambda a_i, \qquad i, j = 1, 2, \dots, n.$$

(B) *There is a real number x such that each $\|a_i x\| \ge \lambda$.*

It is easy to see that $k(n)$ is the maximum λ for which (B) always holds when $a_1, a_2, \dots, a_n$ are arbitrary positive integers. By Lemma 1 we know that we need only to investigate the solubility of the inequalities system in Lemma 1(A). I have done some work in this direction. For related references one may refer to [4]. In [4], remarks, the following problem is proposed.

Problem. *Find the largest α such that given any $\varepsilon > 0$, there exist finitely many real points (l_i, m_i, n_i) $(i = 1, 2, \dots, s)$ satisfying for any three positive numbers a_1, a_2, a_3 with $l_i a_1 + m_i a_2 + n_i a_3 \ne 0$ $(i = 1, 2, \dots, s)$ there exist three integers k_1, k_2, k_3 such that*

$$a_i k_j - a_j k_i \le (1-\lambda) a_j - \lambda a_i, \qquad i, j = 1, 2, 3,$$

where $\lambda = \alpha - \varepsilon$.

In this paper we show how Bacon's quantitative form of Kronecker's theorem in Diophantine approximations implies that $\alpha = \frac{1}{2}$ not only for this particular case but also for general cases. We also given another quantitative form of Kronecker's theorem and use it to prove the following conclusion.

Theorem. *Let $n \ge 4$. Then*

$$\frac{1}{4}\nu(n)^2 = \sup_{A_n(3)} \inf_{t\in[0,1]} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2} \right\|^2,$$

$$\lambda(n) = 2 \sup_{A_n(\frac{1}{2}n+5)} \inf_{t\in[0,1]} \max_{1\le i\le n} \left\| a_v t - \frac{1}{2} \right\|,$$

where $\sup_{A_n(N)}$ *denotes that the supremum is taken over all* n*-tuples of integers* $a_1, a_2, \dots, a_n$ *such that* $(a_1, a_2, \dots, a_n) \in A_n(N)$, *and*

$$A_n(N) = \{(a_1, \dots, a_n) \in \mathbf{N}^n : \text{there exist integers } c_i \ (1 \le i \le n) \text{ with } |c_i| \le N \text{ and not all } c_i \text{ equal to zero such that } c_1a_1 + \cdots + c_na_n = 0\}.$$

2. The answer to the problem

In this section we employ Bacon's quantitative form of Kronecker's theorem to give a complete answer to the problem in the introduction. I thank one of the referees who made me aware of Bacon's result and gave me many suggestions. Let

$$B_n(N) = \{(a_1, \dots, a_n) \in \mathbf{R}^n : \text{there exist integers } c_1, c_2, \dots, c_n \text{ with } 0 < |c_1| + \cdots + |c_n| \le N \text{ such that } c_1a_1 + \cdots + c_na_n = 0\}.$$

Lemma 2 (Bacon [1], p. 784). *If* $(a_1, \dots, a_n) \notin B_n(c(n)N)$, *then there exists a real number* t *such that*

$$\left\| a_it - \frac{1}{2} \right\| < \frac{1}{N}, \qquad i = 1, 2, \dots, n,$$

where

$$c(n) = \frac{1}{2}(n-1)^{3/2}\left(\frac{125}{48}\right)^{(n^3-n)/12}.$$

Now we give the answer to the problem. Noting the fact that

$$\left\| a_it - \frac{1}{2} \right\| = \frac{1}{2} - \|a_it\|,$$

by Lemma 2 we know that for $(a_1, \dots, a_n) \notin B_n(c(n)/\varepsilon)$ there exists a real number t such that

$$\|a_it\| \ge \frac{1}{2} - \varepsilon, \qquad i = 1, 2, \dots, n.$$

Thus by Lemma 1 we know that the largest α in the problem is $1/2$.

3. The proof of the Theorem

Bacon's quantitative form of Kronecker's theorem cannot be employed to prove the theorem for $c(n)$ large enough. In this section we use the method in Bohr and Jessen [2] to prove another quantitative form of Kronecker's theorem, which can be employed to prove the theorem. Let

$$R_n(N) = \{(a_1, \dots, a_n) \in \mathbf{R}^n : \text{there exist integers } c_i \ (1 \le i \le n) \text{ with } |c_i| \le N \text{ and not all } c_i \text{ equal to zero such that } c_1a_1 + \cdots + c_na_n = 0\}.$$

Let $N \ge 2$ be an integer and $b_v = b_v(N)$, $b_{-v} = b_v$ $(1 \le v \le N-1)$ be real numbers satisfying $b_0 = 1$ and

$$G_N(t) \stackrel{\text{def}}{=} \sum_{v=-N+1}^{N-1} b_v e^{ivt} \ge 0 \quad \text{for any real number } t.$$

Lemma 3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. For $(a_1, \ldots, a_n) \notin R_n(N)$ we have*

$$\inf_{t\in\mathbf{R}} \sum_{v=1}^{n} \|a_v t - \alpha_v\|^2 \le \frac{1}{8} n(1 - b_1(N)).$$

Proof. Let

$$F_n(t) = \sum_{v=1}^{n} (e^{-2\pi i(a_v t - \alpha_v)} + e^{2\pi i(a_v t - \alpha_v)}).$$

Since $(a_1, a_2, \ldots, a_n) \notin R_n(N)$, we have

$$\begin{aligned} H_n(t) &= G_N(2\pi(a_1 t - \alpha_1)) \cdots G_N(2\pi(a_n t - \alpha_n)) \\ &= 1 + b_1 F_n(t) + S_n(t), \\ H_n(t)F_n(t) &= 2nb_1 + R_n(t), \end{aligned}$$

where $S_n(t)$ is a trigonometrical polynomial whose exponents are all different from $0, \pm 2\pi a_v$ $(1 \le v \le n)$, $R_n(t)$ is a trigonometrical polynomial whose exponents are all different from 0. Hence

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} H_n(t) F_n(t)\, dt = 2nb_1, \tag{3}$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} H_n(t)\, dt = 1. \tag{4}$$

Again we have $H_n(t) \ge 0$ and

$$\begin{aligned} F_n(t) &= 2\sum_{v=1}^{n} \cos 2\pi(a_v t - \alpha_v) = 2\sum_{v=1}^{n} \cos 2\pi \|a_v t - \alpha_v\| \\ &= 2\sum_{v=1}^{n} (1 - 2\sin^2(\pi \|a_v t - \alpha_v\|)) \qquad (5)\\ &\le 2\sum_{v=1}^{n} (1 - 8\|a_v t - \alpha_v\|^2) \le 2n - 16 \inf_{t\in R} \sum_{v=1}^{n} \|a_v t - \alpha_v\|^2. \end{aligned}$$

By (3), (4) and (5) we have

$$2nb_1 \le 2n - 16 \inf_{t\in\mathbf{R}} \sum_{v=1}^{n} \|a_v t - \alpha_v\|^2.$$

This completes the proof of Lemma 3.

Lemma 4.

$$\lambda(n) \ge \frac{n-1}{n+1}, \qquad \frac{1}{4}\nu(n)^2 \ge \frac{11}{300} n \quad \textit{if } n \ge 4.$$

Proof. That $\lambda(n) \ge \frac{n-1}{n+1}$ is a known result (Cusick [8]). For any positive integers

k, l we have

$$\begin{aligned}
\frac{1}{4}\nu(k+l)^2 &= \sup_{a_1, \dots, a_{k+l}} \inf_{t\in[0,1]} \sum_{i=1}^{k+l} \left\| a_i t - \frac{1}{2}\right\|^2 \\
&= \sup_{\substack{b_1, \dots, b_k \\ c_1, \dots, c_l}} \inf_{t\in[0,1]} \left(\sum_{i=1}^{k} \left\| b_i t - \frac{1}{2}\right\|^2 + \sum_{i=1}^{l} \left\| c_i t - \frac{1}{2}\right\|^2 \right) \\
&\geq \sup_{\substack{b_1, \dots, b_k \\ c_1, \dots, c_l}} \left(\inf_{t\in[0,1]} \sum_{i=1}^{k} \left\| b_i t - \frac{1}{2}\right\|^2 + \inf_{t\in[0,1]} \sum_{i=1}^{l} \left\| c_i t - \frac{1}{2}\right\|^2 \right) \\
&= \sup_{b_1, \dots, b_k} \inf_{t\in[0,1]} \sum_{i=1}^{k} \left\| b_i t - \frac{1}{2}\right\|^2 + \sup_{c_1, \dots, c_l} \inf_{t\in[0,1]} \sum_{i=1}^{l} \left\| c_i t - \frac{1}{2}\right\|^2 \\
&= \frac{1}{4}\nu(k)^2 + \frac{1}{4}\nu(l)^2,
\end{aligned}$$

where a_i, b_i and c_i are positive integers. Thus

$$\begin{aligned}
&\nu(4k)^2 \geq k\nu(4)^2, &\qquad &\nu(4k+1)^2 \geq k\nu(4)^2, \\
&\nu(4k+2)^2 \geq k\nu(4)^2 + \nu(2)^2, &\qquad &\nu(4k+3)^2 \geq k\nu(4)^2 + \nu(3)^2.
\end{aligned}$$

From these inequalities and $\nu(2)^2 = 1/5$ (Cusick [7]), $\nu(3)^2 = 3/7$ (Dumir and Hans-Gill [9]) and

$$\nu(4)^2 \geq 4 \inf_{t\in[0,1]} \left(2\left\| t - \frac{1}{2}\right\|^2 + \left\| 2t - \frac{1}{2}\right\|^2 + \left\| 3t - \frac{1}{2}\right\|^2 \right) = \frac{11}{15},$$

we can derive that $\nu(n)^2 \geq 11n/75$ if $n \geq 4$. In fact I have proved that $\nu(4)^2 = 11/15$ in [5]. This completes the proof of Lemma 4.

The Proof of the Theorem. (A) Let $b_0 = 1$, $b_1 = \sqrt{2}/2$, $b_2 = 1/4$. Then

$$G_3(t) = \sum_{v=-2}^{2} b_v e^{ivt} = \left(\cos t + \frac{\sqrt{2}}{2}\right)^2 \geq 0.$$

By Lemmas 3 and 4 we know that for $(a_1, \dots, a_n) \notin R_n(3)$ we have

$$\inf_{t\in\mathbf{R}} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2}\right\|^2 \leq \frac{1}{8}\left(1 - \frac{\sqrt{2}}{2}\right) n < \frac{11}{300} n \leq \frac{1}{4}\nu(n)^2.$$

Again $\mathbf{N}^n \cap R_n(3) = A_n(3)$. Hence

$$\frac{1}{4}\nu(n)^2 = \sup_{A_n(3)} \inf_{t\in[0,1]} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2}\right\|^2 .$$

(B) Since the Fejer kernel

$$G_N(t) = \sum_{v=-N+1}^{N-1} \frac{N-|v|}{N} e^{ivt} = \frac{1}{N}\left(\frac{\sin(tN/2)}{\sin(t/2)}\right)^2 \geq 0,$$

by Lemmas 3 and 4 we know that for $(a_1, \ldots, a_n) \notin R_n(\frac{1}{2}n+5)$ and $n \geq 4$ we have

$$\inf_{t\in\mathbf{R}} \max_{1\leq v\leq n} \left\| a_v t - \frac{1}{2} \right\|^2 \leq \inf_{t\in\mathbf{R}} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2} \right\|^2 \leq \frac{1}{8}\left(1 - b_1\left(\left[\frac{1}{2}n\right] + 5\right)\right) n$$
$$= \frac{1}{8} \frac{n}{[n/2]+5} < \left(\frac{1}{2}\frac{n-1}{n+1}\right)^2 \leq \left(\frac{1}{2}\lambda(n)\right)^2,$$

where $[x]$ denotes the integral part of x. Again $\mathbf{N}^n \cap R_n(\frac{1}{2}n+5) = A_n(\frac{1}{2}n+5)$. So

$$\lambda(n) = 2 \sup_{A_n(\frac{1}{2}n+5)} \inf_{t\in[0,1]} \max_{1\leq v\leq n} \left\| a_v t - \frac{1}{2} \right\|.$$

Finally we note that in the above we have used the fact that $\|a_v t - \frac{1}{2}\| = \|a_v\{t\} - \frac{1}{2}\|$ for an integer a_v, where $\{t\}$ is a fractional part of t. This completes the proof of the theorem.

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References

1. H. M. Bacon, *An extension of Kronecker's theorem*, Ann. of Math. **35** (1934), 776–786.
2. H. Bohr and B. Jessen, *One more proof of Kronecker's theorem*, J. London Math. Soc. **7** (1932), 274–275.
3. J. W. S. Cassels, *An introduction to Diophantine approximations*, Cambridge Univ. Press, London and New York, 1957.
4. Y. G. Chen, *On a conjecture in Diophantine approximations* III, J. Number Theory **39** (1991), 91–103.
5. ——, *The view-obstruction problem for 4-dimensional spheres*, Amer. J. Math. **116** (1994), 1381–1419.
6. ——, *View-obstruction problems and a generalization in* E^n, Acta Math. Sinica **37** (1994), 551–562. (Chinese)
7. T. W. Cusick, *View-obstruction problems*, Aequationes Math. **9** (1973), 165–170.
8. T. W. Cusick and C. Pomerance, *View-obstruction problems* III, J. Number Theory **19** (1984), 131–139.
9. V. C. Dumir and R. J. Hans-Gill, *View-obstruction problem for 3-dimensional spheres*, Monatsh. Math. **101** (1986), 279–289.
10. V. C. Dumir, R. J. Hans-Gill, and J. B. Wilker, *Contribution to a general theory of view-obstruction problems*, Canad. J. Math. **45** (1993), 517–536.

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