VIEW-OBSTURATION PROBLEMS
AND KRONECKER'S THEOREM

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Abstract. In this paper we show how the quantitative forms of Kronecker's
theorem in Diophantine approximations can be applied to investigate view-
obstruction problems. In particular we answer a question in [Yong-Gao Chen,
91–103].

1. Introduction

In [7] Cusick proposed the view-obstruction problems for n-dimensional ge-
ometry. Here we do not give the original definitions. Instead, we give equiva-
 lent definitions. For details one may refer to Cusick [8] and the author [5]. The
view-obstruction problem for cubes in $E^n$ is to determine the value

\[ \lambda(n) = 2 \sup_{a_1, \ldots, a_n \in [0,1]} \inf_{1 \leq i \leq n} \max \left\{ a_it - \frac{1}{2} \right\}, \]

where the supremum is taken over all n-tuples of positive integers $a_1, a_2, \ldots, a_n$, and $\|x\|$ denotes the distance from $x$ to the nearest integer. The view-
obstruction problem for spheres in $E^n$ is to determine the value

\[ \nu(n) = 2 \left( \sup_{a_1, \ldots, a_n \in [0,1]} \inf \sum_{i=1}^{n} \left\{ a_it - \frac{1}{2} \right\}^2 \right)^{1/2}, \]

where the supremum has the above meaning. We can derive that

\[ \lambda(n) = 1 - 2k(n), \]

where

\[ k(n) = \inf_{a_1, \ldots, a_n \in [0,1]} \sup_{1 \leq i \leq n} \min \|a_it\|. \]

In [7] Cusick conjectured that $k(n) = (n + 1)^{-1}$, and the corresponding value
for $\lambda(n)$ is $(n - 1)/(n + 1)$. Up to now, we know that $\lambda(2) = 1/3$, $\lambda(3) = 1/2$,
$\lambda(4) = 3/5$ (Cusick [7, 8]), $\nu(2) = 1/\sqrt{5}$ (Cusick [7]), $\nu(3) = \sqrt{3/7}$ (Dumir

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and Hans-Gill [9], also see author [4]), \(\nu(4) = \sqrt{11/15}\) (author [5]). In [6] I considered

\[
K_f(n) = \sup_{a_1, \ldots, a_n} \inf_{t \in \mathbb{R}} \sum_{i=1}^{n} f(\|a_i t\|),
\]

for non-negative real-valued functions \(f(x)\) in \([0, 1]\), where the supremum is taken over all \(n\)-tuples of positive real numbers \(a_1, a_2, \ldots, a_n\). There we derived some properties of \(K_f(n)\) and proved that

\[
\lim_{n \to \infty} \frac{\nu(n)^2}{n} = \sup_n \frac{\nu(n)^2}{n},
\]

\[
\nu(n)^2 \leq \frac{n}{3} - \frac{1}{45},
\]

\[
k(n) \geq (2n - 1 - (2n - 3)^{-1})^{-1}, \quad n \geq 5.
\]

In [4] the following lemma is given.

**Lemma 1.** Let \(0 < \lambda \leq \frac{1}{2}\) be a real number and \(a_1, a_2, \ldots, a_n\) be \(n\) positive real numbers. The following statements are equivalent:

(A) There exist \(n\) integers \(k_1, k_2, \ldots, k_n\) such that

\[
a_i k_j - a_j k_i \leq (1 - \lambda)a_j - \lambda a_i, \quad i, j = 1, 2, \ldots, n.
\]

(B) There is a real number \(x\) such that each \(\|a_i x\| \geq \lambda\).

It is easy to see that \(k(n)\) is the maximum \(\lambda\) for which (B) always holds when \(a_1, a_2, \ldots, a_n\) are arbitrary positive integers. By Lemma 1 we know that we need only to investigate the solubility of the inequalities system in Lemma 1(A). I have done some work in this direction. For related references one may refer to [4]. In [4], remarks, the following problem is proposed.

**Problem.** Find the largest \(\alpha\) such that given any \(\varepsilon > 0\), there exist finitely many real points \((l_i, m_i, n_i)\) \((i = 1, 2, \ldots, s)\) satisfying for any three positive numbers \(a_1, a_2, a_3\) with \(l_i a_1 + m_i a_2 + n_i a_3 \neq 0\) \((i = 1, 2, \ldots, s)\) there exist three integers \(k_1, k_2, k_3\) such that

\[
a_i k_j - a_j k_i \leq (1 - \lambda)a_j - \lambda a_i, \quad i, j = 1, 2, 3,
\]

where \(\lambda = \alpha - \varepsilon\).

In this paper we show how Bacon’s quantitative form of Kronecker’s theorem in Diophantine approximations implies that \(\alpha = \frac{1}{2}\) not only for this particular case but also for general cases. We also give another quantitative form of Kronecker’s theorem and use it to prove the following conclusion.

**Theorem.** Let \(n \geq 4\). Then

\[
\frac{1}{4} \nu(n)^2 = \sup_{A_n(1)} \inf_{t \in [0, 1]} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2} \right\|^2,
\]

\[
\lambda(n) = 2 \sup_{A_n(1) + n \leq i \leq n} \inf_{t \in [0, 1]} \max_{1 \leq i \leq n} \left\| a_v t - \frac{1}{2} \right\|,
\]
where \( \sup_{A_n(N)} \) denotes that the supremum is taken over all \( n \)-tuples of integers \( a_1, a_2, \ldots, a_n \) such that \( (a_1, a_2, \ldots, a_n) \in A_n(N) \), and

\[
A_n(N) = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : \text{there exist integers } c_i (1 \leq i \leq n) \text{ with } |c_i| \leq N \text{ and not all } c_i \text{ equal to zero such that } c_1 a_1 + \cdots + c_n a_n = 0\}.
\]

2. THE ANSWER TO THE PROBLEM

In this section we employ Bacon's quantitative form of Kronecker's theorem to give a complete answer to the problem in the introduction. I thank one of the referees who made me aware of Bacon's result and gave me many suggestions. Let

\[
B_n(N) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \text{there exist integers } c_1, c_2, \ldots, c_n \text{ with } 0 < |c_1| + \cdots + |c_n| \leq N \text{ such that } c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0\}.
\]

Lemma 2 (Bacon [1], p. 784). If \((a_1, \ldots, a_n) \notin B_n(c(n)/N)\), then there exists a real number \( t \) such that

\[
\left\| a_i t - \frac{1}{2} \right\| < \frac{1}{N}, \quad i = 1, 2, \ldots, n,
\]

where

\[
c(n) = \frac{1}{2} (n - 1)^{3/2} \left( \frac{125}{48} \right)^{(n^3 - n)/12}.
\]

Now we give the answer to the problem. Noting the fact that

\[
\left\| a_i t - \frac{1}{2} \right\| = \frac{1}{2} - \|a_i t\|,
\]

by Lemma 2 we know that for \((a_1, \ldots, a_n) \notin B_n(c(n)/e)\) there exists a real number \( t \) such that

\[
\|a_i t\| \geq \frac{1}{2} - e, \quad i = 1, 2, \ldots, n.
\]

Thus by Lemma 1 we know that the largest \( \alpha \) in the problem is \( 1/2 \).

3. THE PROOF OF THE THEOREM

Bacon's quantitative form of Kronecker's theorem cannot be employed to prove the theorem for \( c(n) \) large enough. In this section we use the method in Bohr and Jessen [2] to prove another quantitative form of Kronecker's theorem, which can be employed to prove the theorem. Let

\[
R_n(N) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \text{there exist integers } c_i (1 \leq i \leq n) \text{ with } |c_i| \leq N \text{ and not all } c_i \text{ equal to zero such that } c_1 a_1 + \cdots + c_n a_n = 0\}.
\]

Let \( N \geq 2 \) be an integer and \( b_v = b_v(N), \quad b_{-v} = b_v \) \((1 \leq v \leq N - 1)\) be real numbers satisfying \( b_0 = 1 \) and

\[
G_N(t) \overset{\text{def}}{=} \sum_{v=-N+1}^{N-1} b_v e^{itv} \geq 0 \quad \text{for any real number } t.
\]
Lemma 3. Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be real numbers. For \((a_1, \ldots, a_n) \not\in R_n(N)\) we have
\[
\inf_{t \in \mathbb{R}} \sum_{v=1}^{n} ||a_v t - \alpha_v||^2 \leq \frac{1}{8} n(1 - b_1(N)).
\]

Proof. Let
\[
F_n(t) = \sum_{v=1}^{n} (e^{-2\pi i (a_v t - \alpha_v)} + e^{2\pi i (a_v t - \alpha_v)}).
\]

Since \((a_1, a_2, \ldots, a_n) \not\in R_n(N)\), we have
\[
H_n(t) = G_N(2\pi (a_1 t - \alpha_1)) \cdots G_N(2\pi (a_n t - \alpha_n))
= 1 + b_1 F_n(t) + S_n(t),
\]
\[
H_n(t) F_n(t) = 2nb_1 + R_n(t),
\]
where \(S_n(t)\) is a trigonometrical polynomial whose exponents are all different from 0, \(\pm 2\pi a_v\) \((1 \leq v \leq n)\), \(R_n(t)\) is a trigonometrical polynomial whose exponents are all different from 0. Hence
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} H_n(t) F_n(t) dt = 2nb_1 \tag{3}
\]
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} H_n(t) dt = 1. \tag{4}
\]

Again we have \(H_n(t) \geq 0\) and
\[
F_n(t) = 2 \sum_{v=1}^{n} \cos 2\pi (a_v t - \alpha_v) = 2 \sum_{v=1}^{n} \cos 2\pi ||a_v t - \alpha_v||
= 2 \sum_{v=1}^{n} (1 - 2 \sin^2(\pi ||a_v t - \alpha_v||))
\leq 2 \sum_{v=1}^{n} (1 - 8 ||a_v t - \alpha_v||^2) \leq 2n - 16 \inf_{t \in \mathbb{R}} \sum_{v=1}^{n} ||a_v t - \alpha_v||^2.
\]

By (3), (4) and (5) we have
\[
2nb_1 \leq 2n - 16 \inf_{t \in \mathbb{R}} \sum_{v=1}^{n} ||a_v t - \alpha_v||^2.
\]

This completes the proof of Lemma 3.

Lemma 4.
\[
\lambda(n) \geq \frac{n - 1}{n + 1}, \quad \frac{1}{4} \nu(n)^2 \geq \frac{11}{300} n \quad \text{if } n \geq 4.
\]

Proof. That \(\lambda(n) \geq \frac{n - 1}{n + 1}\) is a known result (Cusick [8]). For any positive integers
$k, l$ we have

$$\frac{1}{4} \nu(k + l)^2 = \sup_{a_1, \ldots, a_k, \ell \in [0, 1]} \inf_{t \in [0, 1]} \sum_{i=1}^{k+l} \left\| a_it - \frac{1}{2} \right\|^2$$

$$= \sup_{b_1, \ldots, b_k} \inf_{c_1, \ldots, c_l} \left( \sum_{i=1}^{k} \left\| b_it - \frac{1}{2} \right\|^2 + \sum_{i=1}^{l} \left\| c_it - \frac{1}{2} \right\|^2 \right)$$

$$\geq \sup_{b_1, \ldots, b_k} \left( \inf_{t \in [0, 1]} \sum_{i=1}^{k} \left\| b_it - \frac{1}{2} \right\|^2 + \inf_{t \in [0, 1]} \sum_{i=1}^{l} \left\| c_it - \frac{1}{2} \right\|^2 \right)$$

$$= \sup_{b_1, \ldots, b_k} \left( \sum_{i=1}^{k} \left\| b_it - \frac{1}{2} \right\|^2 + \sup_{c_1, \ldots, c_l} \left( \inf_{t \in [0, 1]} \sum_{i=1}^{l} \left\| c_it - \frac{1}{2} \right\|^2 \right) \right)$$

$$= \frac{1}{4} \nu(k)^2 + \frac{1}{4} \nu(l)^2,$$

where $a_i, b_i$ and $c_i$ are positive integers. Thus

$$\nu(4k)^2 \geq k \nu(4)^2,$$

$$\nu(4k + 1)^2 \geq k \nu(4)^2,$$

$$\nu(4k + 2)^2 \geq k \nu(4)^2 + \nu(2)^2,$$

$$\nu(4k + 3)^2 \geq k \nu(4)^2 + \nu(3)^2.$$

From these inequalities and $\nu(2)^2 = 1/5$ (Cusick [7]), $\nu(3)^2 = 3/7$ (Dumir and Hans-Gill [9]) and

$$\nu(4)^2 \geq 4 \inf_{t \in [0, 1]} \left( 2 \left\| t - \frac{1}{2} \right\|^2 + 2 \left\| 2t - \frac{1}{2} \right\|^2 + 3 \left\| 3t - \frac{1}{2} \right\|^2 \right) \geq \frac{11}{15},$$

we can derive that $\nu(n)^2 \geq 11n/75$ if $n \geq 4$. In fact I have proved that $\nu(4)^2 = 11/15$ in [5]. This completes the proof of Lemma 4.

The Proof of the Theorem. (A) Let $b_0 = 1, b_1 = \sqrt{2}/2, b_2 = 1/4$. Then

$$G_3(t) = \sum_{v=-2}^{2} b_v e^{ivt} = \left( \cos t + \frac{\sqrt{2}}{2} \right)^2 \geq 0.$$

By Lemmas 3 and 4 we know that for $(a_1, \ldots, a_n) \notin R_n(3)$ we have

$$\inf_{t \in [0, 1]} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2} \right\|^2 \leq \frac{1}{8} \left( 1 - \frac{\sqrt{2}}{2} \right) n < \frac{11}{300} n \leq \frac{1}{4} \nu(n)^2.$$

Again $N^n \cap R_n(3) = A_n(3)$. Hence

$$\frac{1}{4} \nu(n)^2 = \sup_{A_n(3)} \inf_{t \in [0, 1]} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2} \right\|^2.$$

(B) Since the Fejer kernel

$$G_N(t) = \sum_{v=-N+1}^{N-1} \frac{N - |v|/N}{N} e^{ivt} = \frac{1}{N} \left( \frac{\sin(tN/2)}{\sin(t/2)} \right)^2 \geq 0.$$
by Lemmas 3 and 4 we know that for \( (a_1, \ldots, a_n) \not\in R_n(\frac{1}{2}n + 5) \) and \( n \geq 4 \) we have

\[
\inf_{t \in \mathbb{R}} \max_{1 \leq v \leq n} \left\| a_v t - \frac{1}{2} \right\|^2 \leq \inf_{t \in \mathbb{R}} \sum_{v=1}^{n} \left\| a_v t - \frac{1}{2} \right\|^2 \leq \frac{1}{8} \left( 1 - b_1 \left( \left\lfloor \frac{1}{2}n \right\rfloor + 5 \right) \right) n
\]

\[
= \frac{1}{8} \frac{n}{[n/2] + 5} < \left( \frac{1}{2} \frac{n - 1}{n + 1} \right)^2 \leq \left( \frac{1}{2} \lambda(n) \right)^2 ,
\]

where \( [x] \) denotes the integral part of \( x \). Again \( N^n \cap R_n(\frac{1}{2}n + 5) = A_n(\frac{1}{2}n + 5) \).

So

\[
\lambda(n) = 2 \sup_{A_n(\frac{1}{2}n + 5)} \inf_{t \in [0,1]} \max_{1 \leq v \leq n} \left\| a_v t - \frac{1}{2} \right\| .
\]

Finally we note that in the above we have used the fact that \( \left\| a_v t - \frac{1}{2} \right\| = \left\| a_v \{t\} - \frac{1}{2} \right\| \) for an integer \( a_v \), where \( \{t\} \) is a fractional part of \( t \). This completes the proof of the theorem.

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**REFERENCES**


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