PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 11, November 1995

n-LAPLACIAN IN $\mathcal{H}^1_{\text{loc}}$ DOES NOT LEAD TO REGULARITY

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(Communicated by Barbara Lee Keyfitz)

ABSTRACT. It is well known that in two space dimensions, if a solution to Poisson's equation has right-hand side in $\mathcal{H}^1_{\text{loc}}$, then this solution is actually continuous. The corresponding result for n-Laplacians is shown to be false for $n \geq 3$; we construct two examples with right-hand sides in $\mathcal{H}^1_{\text{loc}}(\mathfrak{R}^n)$ such that the corresponding solutions to the n-Laplacian are unbounded in the first case, and bounded but discontinuous in the second.

1. INTRODUCTION

It is well known that, although $\Delta u \in L^p$ for $1 < p < \infty$ implies that $u \in W^{2,p}$, and thus via the Sobolev embedding theorem that (for $n/2 < p < n$) u is in a Hölder class, the corresponding result for $p = 1$ is untrue: $\Delta u \in L^1$ does not imply that u is even continuous. Instead, one needs to assume more for the right-hand side. A sufficient condition on the right-hand side in order to assure continuity of the solution takes a particularly simple form in two dimensions: if $\Delta u \in \mathcal{H}^1(\mathfrak{R}^2)$, then by applying two Riesz transforms, we know that $D_i D_j u \in \mathcal{H}^1$, in turn implying that $u \in W^{2,1}(\mathfrak{R}^2)$, and then a particularly simple embedding theorem gives us the absolute continuity of u. Another proof of the continuity of u follows by convolving the right-hand side with the fundamental solution to Poisson's equation in two dimensions, $\log(1/|x|)$, which is in $(\mathcal{H}^1)^* = \text{BMO}$. Boundedness of the solution is immediate. Continuity comes from the fact that the translation operator $T_a[f](x) = f(x+a)$ is continuous in the strong topology of $\mathcal{H}^1$ and Poisson's equation is translation invariant. Without much difficulty, these results can be formulated in terms of the corresponding local spaces. This result was key to the proof of the regularity of weak harmonic immersions from two-dimensional domains into the 2-sphere by Hélein ([5], [6]) and into general Riemannian manifolds, and to the extension of Evans for partial regularity of stationary harmonic maps from n-dimensional domains into the m-sphere ([3]) as well as that of Bethuel

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Received by the editors October 29, 1993 and, in revised form, April 8, 1994.
1991 *Mathematics Subject Classification*. Primary 35J60, 35D10; Secondary 42B30, 58E20.
This work was completed while the author was visiting the Institute for Mathematics and its Applications, University of Minnesota, 514 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455. This work was partially supported by a National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship.

for partial regularity of stationary harmonic maps with general target manifold ([1]).

Partially with the hope of extending Hélein's results to n-harmonic maps and partially for the sake of studying the simplest of degenerate elliptic equations, the question arises of whether the n-Laplacian, with right-hand side in $\mathscr{H}^1_{\mathrm{loc}}$,

$$\Delta_n u = \operatorname{div}(|\nabla u|^{n-2}\nabla u) = f \in \mathscr{H}^1_{\mathrm{loc}}$$

has continuous solution u. The regularity of such degenerate elliptic equations has been studied extensively by Uhlenbeck [12], Tolksdorff [11], and DiBenedetto-Manfredi [2], among others, particularly in the case of zero or Hölder continuous right-hand sides. The n-Laplacian in $\mathfrak{R}^n$ scales the same way that the Laplacian does in $\mathfrak{R}^2$: letting $\tilde{u}(x) = u(x/\varepsilon)$, $\Delta_n \tilde{u}(x) = \operatorname{div}(|\nabla\tilde{u}|^{n-2}(x)\tilde{u}(x)) = (1/\varepsilon)^n \Delta_n u(x/\varepsilon)$. Thus, if we predict a bound of the form

$$\|u\|_Y \leq C\|\Delta_n u\|_X,$$

with X and Y Banach spaces, where $Y = L^\infty$ or C^0, then we expect X to scale like L^1 or $\mathscr{H}^1$. We shall show that $\mathscr{H}^1$ or $\mathscr{H}^1_{\mathrm{loc}}$ is not sufficient for either of these bounds. Fortunately, the regularity results for the special case of n-harmonic mappings into the m-sphere, analogous to those which Hélein proved, do not rely upon this sufficiency; Mou and Yang ([8]) have used more subtle analysis of the equations together with local $\mathscr{H}^1_{\mathrm{loc}}$ bounds, similar to those employed by Evans and Bethuel, in order to extend Hélein's results to the case of stationary n-harmonic maps or those satisfying the monotonicity formula. And, in the special case of the sphere, weakly n-harmonic maps automatically satisfy the monotonicity formula. These counterexamples below show that Mou and Yang's results may not be generalizable to the case of regularity of weakly n-harmonic maps into a general Riemannian manifold (i.e., Hélein's method of proof cannot be applied without some completely new idea). (After having written this note, Fang-Hua Lin (personal communication) reported that weakly n-harmonic maps are regular in spite of this obstacle.)

Let us recall the definitions of $\mathscr{H}^1$ and $\mathscr{H}^1_{\mathrm{loc}}$. Let $\eta \geq 0$ be smooth with compact support and $\int \eta = 1$. Let $\eta_r(x) = (1/r^n)\eta(x/r)$. Define

$$f^*(x) = \sup_{0<r<\infty} |\eta_r * f(x)|.$$

We define $f \in \mathscr{H}^1$ as equivalent to $f^* \in L^1$. Similarly, we define

$$f^{**}(x) = \sup_{0<r\leq 1} |\eta_r * f(x)|.$$

Then we say that $f \in \mathscr{H}^1_{\mathrm{loc}}$ if and only if $f^{**} \in L^1_{\mathrm{loc}}$. The space $\mathscr{H}^1$ is frequently used as a replacement for L^1 in problems having to do with singular integral operators and partial differential equations. The space $\mathscr{H}^1_{\mathrm{loc}}$ is one of the many possible replacements for L^1_{loc} (see Evans-Müller [4]). A result of Stein's ([10]) relating $\mathscr{H}^1_{\mathrm{loc}}$ to higher integrability states that if we assume that $f \in L^1_{\mathrm{loc}}$ and if $|f|\log(|f|+2) \in L^1_{\mathrm{loc}}$, then $f \in \mathscr{H}^1_{\mathrm{loc}}$ (see, e.g., Semmes [9]); if f is non-negative, the converse is true.

2. Counterexamples

Before constructing our counterexamples, we will first rewrite the problem in the radial case:

$$\Delta_n u(r) = \frac{1}{r^{n-1}}(|u'|^{n-2}u'r^{n-1})' = f(r).$$

So, $f \geq 0$ is in $\mathscr{H}^1_{\text{loc}}$ if and only if

$$\int_0^1 f(r)\log(f(r)+2)r^{n-1}\,dr < \infty.$$

We will be interested in showing that elements of a particular class of functions are in $\mathscr{H}_1^{\text{loc}}$. Consider a function of the form

$$g = \frac{1}{r^n}\log^\beta(1/r)G(\log^\alpha(1/r))\chi_{[0,\delta]}(r), \tag{1}$$

where $|G| \leq 1$ may be a periodic function or a constant, $\beta < -2$, and $\delta > 0$ is small. We claim that all such functions are in $\mathscr{H}^1_{\text{loc}}$. Indeed,

$$\begin{aligned}
&\frac{1}{r^n}|\log^\beta(1/r)|\,|G(\log^\alpha(1/r))|\log\left(2+|G(\log^\alpha(1/r))|\frac{1}{r^n}|\log^\beta(1/r)|\right)\\
&\leq \frac{1}{r^n}|\log^\beta(1/r)|\log\left(2+\frac{1}{r^n}|\log^\beta(1/r)|\right)\\
&\leq C\frac{1}{r^n}|\log^\beta(1/r)|\left|\log\left(\frac{1}{r^n}|\log^\beta(1/r)|\right)\right|\\
&\leq C'\frac{1}{r^n}|\log^{\beta+1}(1/r)|,
\end{aligned}$$

where in the second and third lines, we assume that $r < \delta < 1$. We note that this upper bound is locally integrable if $\beta < -2$, since

$$\int_0^\delta \frac{1}{r^n}|\log^{\beta+1}(1/r)| = \int_{-\infty}^{\log(\delta)} s^{\beta+1}\,ds < \infty,$$

for $\beta < -2$.

For our first example, we take $f(r) = r^{-n}\log^\beta(1/r)\chi_{[0,\delta]}(r)$ with $\beta < -2$ and $0 < \delta < 1$, which is in $\mathscr{H}^1_{\text{loc}}$, since it is of the form (1). Solving the equations explicitly

$$\frac{1}{r^{n-1}}(r^{n-1}(u')^{n-1}) = \frac{1}{r^n}\log^\beta(1/r),$$

we see that $u(r) = \log^\alpha(1/r)$ for $\alpha = (n+\beta)/(n-1)$ ($\alpha > 0$ if $-n < \beta < -2$) is a solution and $u \notin L^\infty_{\text{loc}}$.

For the second counterexample, where we seek $u \in L^\infty$ and $\Delta_n u \in \mathscr{H}^1_{\text{loc}}(\mathfrak{R}^n)$ but $u \notin C^0$, the right-hand side cannot be quite as simple. In the radial case, two simple integrations of the ODE's show that, if the right-hand side is of fixed sign, boundedness implies continuity at the origin. Thus, if we wish to remain in the radial framework, we must consider right-hand sides of variable sign. We also note that in the case of non-negative n-superharmonic functions, boundedness implies continuity by a result of Malý and Kilpeläinen ([7]).

Let $u = \sin(\log^\alpha(1/r))$. Clearly $u \in L^\infty$ but is not continuous at the origin. We claim that $\Delta_n u$ is a $\mathscr{H}^1_{\text{loc}}$ function. As a matter of routine calculation, we find that

$$\begin{aligned}
&\left|\frac{1}{r^{n-1}}(r^{n-1}|u'|^{n-2}u')'\right| \\
&\quad \leq \left(\left|(n-1)\alpha(\cos\log^\alpha(1/r))^{n-2}\sin\log^\alpha(1/r)(\log(1/r))^{n(\alpha-1)}\right|\right. \\
&\qquad \left. + \left|(n-1)(\alpha-1)(\cos\log^\alpha(1/r))^{n-1}(\log(1/r))^{(n-1)(\alpha-1)-1}\right|\right)\frac{1}{r^n} \\
&\quad = f_1(r) + f_2(r).
\end{aligned}$$

If we choose $0 < \alpha < 1 - 2/n$, then we see that the powers of the logarithmic terms in both f_1 and f_2 are less than -2, and thus f_1, f_2 are of the form (1) and, therefore, are in $\mathscr{H}^1_{\text{loc}}$.

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