PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 11, November 1995

# THE GENERAL LOCAL FORM OF AN ANALYTIC MAPPING INTO THE SET OF IDEMPOTENT ELEMENTS OF A BANACH ALGEBRA

J.-PH. LABROUSSE

(Communicated by Palle E. T. Jorgensen)

*Dedicated to the memory of Professor Jacques Morgenstern*

ABSTRACT. This paper gives a general formula which describes any analytical mapping of a suitably small open neighborhood $U$ in $\mathbb{C}$ into the set of idempotent elements of any complex Banach algebra $\mathbf{B}$ and an application of this formula to the case when $\mathbf{B}$ is a Calkin algebra.

Let $\mathbf{B}$ be a complex Banach algebra, for any $X, Y \in \mathbf{B}$ let $[X, Y] = XY - YX$, let $U$ be an open subset of $\mathbb{C}$ (which we can assume without loss of generality to contain 0), let $F(z)$ be an analytic mapping of $U$ into $\mathbf{B}$ and let $P \in \mathbf{B}$ be idempotent $(P \neq 0)$. Then the main result of this paper is the following:

**Theorem 1.** *If* $[F(0), P] = 0$ *there exists an open neighborhood* $V$ *of* 0 *in* $\mathbb{C}$ *and two analytic mappings* $P(z)$, $R(z)$ *of* $V$ *in* $\mathbf{B}$ *such that*:
(i) $P(0) = P$ *and for all* $z \in V$, $P(z)$ *is idempotent,*
(ii) *for all* $z \in V$, $[R(z), P] = 0$,
(iii) *for all* $z \in V$, $F(z) = P(z) + R(z)$.
*Moreover, in a small enough neighborhood of* 0, *the pair of mappings* $P(z)$, $R(z)$ *is uniquely determined by properties* (i) *to* (iii).

**Definition 2.** Let $z \in \mathbb{C}$, with $0 \leq |z| \leq \frac{1}{4}$ and set $f(z) = \frac{1}{2} - \sqrt{\frac{1}{4} - z}$. If $n = 1, 2, \ldots$ define $c_n$ (the $n$th Catalan number, cf. [1], [3]) as follows:

$$c_n = \frac{f^{(n)}(0)}{n!}.$$

Clearly:

$$c_n = \frac{(2n-2)!}{n!(n-1)!} = [2n-(2n-1)]\frac{(2n-2)!}{n!(n-1)!} = 2\frac{(2n-2)!}{(n-1)!(n-1)!} - \frac{(2n-1)!}{n!(n-1)!} \in \mathbb{N}.$$

**Lemma 3** (see [3] for a slightly weaker version of this lemma). *Using the notation introduced above we have*:

Received by the editors March 14, 1994 and, in revised form, May 23, 1994.
1991 *Mathematics Subject Classification*. Primary 47A56, 46H30.

(i) *The series* $\sum_{j=1}^{\infty} c_j z^j$ *is uniformly convergent to* $f(z)$ *in* $0 \le |z| \le \frac{1}{4}$.
(ii) $\forall n \in \mathbb{N}^*$, $c_{n+1} = \sum_{j=1}^{n} c_j c_{n+1-j}$.

*Proof*. (i) Let us show first that

$$\sum_{j=1}^{\infty} c_j \left(\frac{1}{4}\right)^j \le \frac{1}{2}. \tag{1}$$

If $n = 1, 2, \ldots$ set $S_n = \sum_{j=1}^{n} c_j (\frac{1}{4})^j$. Then:

$$\begin{aligned} S_{n+1} &+ \left(\frac{1}{4}\right)^{(n+1)} \frac{(2n+1)!}{n!(n+1)!} \\ &= S_n + \left(\frac{1}{4}\right)^{(n+1)} \left(\frac{(2n)!}{n!(n+1)!} + \frac{(2n+1)!}{n!(n+1)!}\right) \\ &= S_n + \left(\frac{1}{4}\right)^{(n+1)} \frac{(2n+2)(2n)!}{n!(n+1)!} = S_n + \left(\frac{1}{4}\right)^n \frac{(2n-1)!}{n!(n-1)!}. \end{aligned}$$

Hence $S_n + (\frac{1}{4})^n \frac{(2n-1)!}{n!(n-1)!}$ is independent of $n$ and since for $n = 1$ its value is $\frac{c_1}{4} + \frac{1}{4} = \frac{1}{2}$, (1) follows at once, and this implies that for $0 \le |z| < \frac{1}{4}$ the function represented by the series is analytic and coincides with $f(z)$. By continuity this still holds for $|z| = \frac{1}{4}$ so that (i) is established and (ii) is easily derived from the fact that $(f(z))^2 = \frac{1}{4} + \frac{1}{4} - z - 2(\frac{1}{2})\sqrt{\frac{1}{4} - z} = f(z) - z$.

**Lemma 4.** *Let* $A \in \mathbf{B}$. *Set* $A_1 = [A, P]$, $A_2 = [A_1, P]$. *Then*:
(a) $[A_2, P] = A_1$.
(b) $PA_1 + A_1P = A_1$.
(c) $PA_2 + A_2P = A_2$.
(d) $P$ *commutes with* $A_1^2$, *with* $A_2^2$ *and with* $A_1A_2$.
(e) $A_1A_2 + A_2A_1 = 0$.
(f) $A_1^2 + A_2^2 = 0$.

*Proof*. (a), (b) and (c) are established by direct computation from the fact that $A_1 = AP - PA$ and that $A_2 = AP - PA - 2PAP$.

(d) is an obvious consequence of (b) and (c).

To prove (e) notice that $A_1A_2 + A_2A_1 = A_1(A_1P - PA_1) + (A_1P - PA_1)A_1 = A_1^2P - PA_1^2 = 0$, using (d). Finally (f) follows from the fact that $A_1^2 + A_2^2 = A_1^2 + (A_1P - PA_1)^2 = A_1^2 + A_1PA_1P + PA_1PA_1 - A_1PA_1 - PA_1^2P = A_1^2 - (I - P)A_1^2 - PA_1^2 = 0$.

**Lemma 5.** *Let* $A \in \mathbf{B}$ *be such that* $\|A_1\| \le 1/(4\|P\|)$. *Then*

$$Q = P + A_2 + \sum_{j=1}^{\infty} c_j A_1 A_2^{2j-1} \quad \textit{is an idempotent element of } \mathbf{B}.$$

*Proof*. Under the given hypothesis $\|A_2\| \le 2\|A_1\|\,\|P\| \le \frac{1}{2}$. Hence using

Lemma 3, the series that defines $Q$ is uniformly convergent. Furthermore:

$$\begin{aligned}
Q^2 &= P + PA_2 + \sum_{j=1}^{\infty} c_j P A_1 A_2^{2j-1} + A_2 P + A_2^2 + \sum_{j=1}^{\infty} c_j A_2 A_1 A_2^{2j-1} \\
&\quad + \sum_{j=1}^{\infty} c_j A_1 A_2^{2j-1} P + \sum_{j=1}^{\infty} c_j A_1 A_2^{2j} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_j c_k A_1 A_2^{2j-1} A_1 A_2^{2k-1} \\
&= P + A_2 + 2\sum_{j=1}^{\infty} c_j P A_1 A_2^{2j-1} + A_2^2 \\
&\quad + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_j c_k A_1 A_2^{2j-1} A_1 A_2^{2k-1} \quad \text{(using (b), (c) and (e))} \\
&= P + A_2 + 2PA_1 \sum_{j=1}^{\infty} c_j A_2^{2j-1} + A_2^2 - \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} c_{i+1-k} c_k A_1^2 A_2^{2i} \\
&= P + A_2 + (A_1 - A_2)\sum_{j=1}^{\infty} c_j A_2^{2j-1} + A_2^2 \\
&\quad + \sum_{i=1}^{\infty} c_{i+1} A_2^{2i+2} \quad \text{(using (f) and Lemma 3)} \\
&= P + A_2 + \sum_{j=1}^{\infty} c_j A_1 A_2^{2j-1} = Q.
\end{aligned}$$

We are now in a position to prove Theorem 1.

*Existence of the decomposition.* Let $F(z)$ be analytic mapping of $U$ into $\mathbf{B}$ such that $[F(0), P] = 0$. Then there exists a neighborhood $V \subseteq U$ such that $\forall z \in V$, $\|[F(z), P]\| < 1/(4\|P\|)$ and therefore the mapping of $V$ into $\mathbf{B}$ defined by

$$(2) \qquad P(z) = P + [[F(z), P], P] + \sum_{j=1}^{\infty} c_j [F(z), P][[F(z), P], P]^{2j-1}$$

is analytic on $V$. Note that $P(0) = P$ and that $\forall z \in V$, $P(z)$ is idempotent. Then, if $R(z) = F(z) - P(z)$,

$$\begin{aligned}
[R(z), P] &= [F(z), P] - [P(z), P] \\
&= [F(z), P] - [F(z), P] \\
&\quad + \sum_{j=1}^{\infty} c_j \{[F(z), P][[F(z), P], P]^{2j-1} P \\
&\qquad - P[F(z), P][[F(z), P], P]^{2j-1}\} \\
&= 0
\end{aligned}$$

using Lemma 4. Hence the existence of the decomposition is proved.

*Uniqueness of the decomposition.* Assume that $F(z) = P_1(z) + R_1(z) = P_2(z) + R_2(z)$ with $P_1(0) = P_2(0) = P$. Set $D(z) = P_1(z) - P_2(z) = R_1(z) - R_2(z)$.

Then $D(z)$ commutes with $P$ and $D(z) = P_1^2(z) - P_2^2(z) = D(z)P_1(z) + P_2(z)D(z)$. Let us show by induction that

$$\forall n \in \mathbb{N}, \quad D^{(n)}(0) = 0. \tag{3}$$

For $n = 0$ we see that $D(0) = P_1(0) - P_2(0) = 0$. Assume that (3) has been proved for $n = 0, 1, \dots, r$. Then using Leibniz's formula we get

$$\begin{aligned} D^{(r+1)}(0) &= \sum_{j=0}^{r+1} \binom{r+1}{j} \{D^{(r+1-j)}(0)P_1^{(j)}(0) + P_2^{(j)}(0)D^{(r+1-j)}(0)\} \\ &= D^{(r+1)}(0)P + PD^{(r+1)}(0), \end{aligned}$$

using the induction hypothesis. So $D^{(r+1)}(0) = 2PD^{(r+1)}(0)$, whence $PD^{(r+1)}(0) = 2PD^{(r+1)}(0)$ so that $PD^{(r+1)}(0) = 0$, therefore $D^{(r+1)}(0) = 0$ and (3) is proved. Hence $D(z) = 0$, which concludes the proof of the theorem.

*Remark.* (2) gives the general local form of any analytic mapping of a neighborhood of 0 into the set of idempotent elements of $\mathbf{B}$. Indeed suppose that $P$ is such a mapping defined on a neighborhood $U$ of 0 into $\mathbf{B}$ and set $P = P(0)$. Then $P(0)P = PP(0)$ and by Theorem 1 there exists a neighborhood $V$ of $\{0\}$ such that

$$Q(z) = P + [[P(z), P], P] + \sum_{j=1}^{\infty} c_j [P(z), P][[P(z), P], P]^{2j-1}$$

is an analytical mapping of a neighborhood $V$ of 0 into the set of the idempotent elements of $\mathbf{B}$ and we have

$$P(z) = Q(z) + R(z) = P(z) + 0$$

and since the decomposition is unique, it follows that $Q(z) = P(z)$ and consequently that $P(z)$ is locally of the form given by (2).

*Application.* Let $H$ be a complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators on $H$ and $K(H)$ the closed bilateral ideal constituted by the compact operators of $L(H)$. Then $A(H) = L(H)/K(H)$ is the Calkin algebra associated to $L(H)$. Let $\pi$ denote the natural mapping of $L(H)$ onto $A(H)$.

**Theorem 2.** *Let $U$ be a neighborhood of 0 in $\mathbb{C}$, and let $p(z)$ be an analytic mapping of $U$ into the space of the idempotent elements of $A(H)$. Then there exists a neighborhood $V$ of 0 in $\mathbb{C}$ and an analytic mapping $P(z)$ of $V$ into the set of the idempotent elements of $L(H)$ such that*

$$\forall z \in V, \quad p(z) = \pi(P(z)).$$

*Proof.* By assumption $\forall z \in U$, $p(z) = \sum_{j=0}^{\infty} p_j z^j$ where $p_0$ is idempotent.

According to [2], Proposition 7, there exists $F_0 \in L(H)$, idempotent, such that $\pi(F_0) = p_0$. Furthermore for every $j \geq 1$ there exists $F_j \in L(H)$ such that $\pi(F_j) = p_j$ with $\|F_j\| \leq 2\|p_j\|$ (this is an obvious consequence of the definition of the norm in $A(H)$). Hence $F(z) = \sum_{j=0}^{\infty} F_j z^j$ converges and is analytic in some neighborhood $V$ of 0 and we have $\forall z \in V$, $\pi(F(z)) = p(z)$. But $F(0) = F_0$. So using Theorem 1, taking $P = F_0$, $F(z) = P(z) +$

$R(z)$, and therefore $p(z) = \pi(P(z)) + \pi(R(z))$ where $\pi(P(z))$ is idempotent with $\pi(P(0)) = p_0$ and $\pi(R(z))$ commutes with $p_0$. Hence, because of the uniqueness of the decomposition, $\pi(R(z)) = 0$ and we have shown that there exists an idempotent-valued mapping $P(z)$, analytic on $V$, such that

$$\forall z \in V, \quad p(z) = \pi(P(z)).$$

## References

1. E. Catalan, *Note sur une équation aux différences finies*, J. Math. Pure Appl. **3** (1838), 508–517.
2. P. De La Harpe, *Initiation à l'algèbre de Calkin*, Lecture Notes in Math, vol. 725, Springer-Verlag, Berlin and New York, 1979, pp. 180–219.
3. J. H. van Lint and R. M. Wilson, *A course in combinatorics*, Cambridge Univ. Press, London and New York, 1992, pp. 116–117.

Laboratoire J. A. Dieudonné, URA CNRS 168, Université de Nice Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France
*E-mail address*: labro@math.unice.fr