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ON COMPACT KAEHLER SUBMANIFOLDS IN $\mathbb{C}P^{n+p}$ WITH NONNEGATIVE SECTIONAL CURVATURE

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ABSTRACT. A complete classification for nonnegatively curved compact Kaehler submanifolds M^n in $\mathbb{C}P^{n+p}$ with $p < n$ is given, so that a conjecture of K. Ogiue is resolved partially.

1. INTRODUCTION

Let $\mathbb{C}P^{n+p}$ be an $(n+p)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. There are a number of conjectures for Kaehler submanifolds M^n in CP^{n+p} suggested by K. Ogiue ([1]); some have been resolved under a suitable topological restriction (e.g., M^n is complete) (cf. [2, 3, 4, 5]). In this direction, one of the open problems so far is as follows.

Conjecture (K. Ogiue). Let M^n be a complete Kaehler submanifold immersed in $\mathbb{C}P^{n+p}$. If the sectional curvature $K > 0$ for M^n and if $p < n(n+1)/2$, then M^n is totally geodesic in $\mathbb{C}P^{(n+p)}$.

In the case that $p = 1$, i.e., M^n is a complex hypersurface in $\mathbb{C}P^{n+1}$, it was pointed out in [1] that this conjecture is true if either $n \geq 4$ or $n \geq 2$ and M^n is an imbedded hypersurface in $\mathbb{C}P^{n+1}$. Recently, the case that $n \geq 2$ and M^n is an immersed hypersurface in $\mathbb{C}P^{n+1}$ was studied by W. Sheng ([6]). In the case of higher codimension, in general, a stronger pinching condition for the sectional curvature K of M^n is required, so that M^n is indeed compact; for example, see [3, 4].

In this note we would like to consider the case that $K \geq 0$ and $p < n$, so that the above conjecture is resolved partially. The main result of the present paper is the following

Theorem. *Let M^n be an $n(\geq 2)$-dimensional compact Kaehler submanifold immersed in $\mathbb{C}P^{n+p}$. Then M^n has nonnegative sectional curvature and $p < n$ if and only if M^n is either totally geodesic in $\mathbb{C}P^{n+p}$ or an imbedded submanifold*

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congruent to the standard imbedding of one of the following submanifolds:

Submanifolds	n	p
$M_1 = Q^n$	n	1
$M_2 = \mathbb{C}P^{n-1} \times \mathbb{C}P^1$	n	$n-1$
$M_3 = U(s+2)/U(s) \times U(2)$	$2s$ $(s = 3, 4)$	$s(s-1)/2$
$M_4 = \mathrm{SO}(10)/U(5)$	10	5
$M_5 = E_6/\mathrm{Spin}(10) \times T$	16	10

where n *and* p *are the complex dimension and the full complex codimension of submanifolds.*

As a direct consequence, we have

Corollary 1. *Under the same hypothesis as in the theorem, if the sectional curvature* $K > 0$ *for* M^n, *then* M^n *must be totally geodesic in* $\mathbb{C}P^{n+p}$.

Clearly, it contains the result of [6]. In other words, the above conjecture of K. Ogiue is true for compact M^n in $\mathbb{C}P^{n+p}$ and for $p < n$. It is known ([1], Proposition 6.12) that the completeness for negatively curved Kaehler hypersurfaces in $\mathbb{C}P^{n+1}$ implies the compactness. So, from Theorem we obtain also

Corollary 2. *If* M^n, $n \geq 2$, *is a complete nonnegatively curved Kaehler hypersurface in* $\mathbb{C}P^{n+1}$, *then* M^n *is either totally geodesic or congruent to either* Q^n *or* $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Furthermore, by combining Corollary 1 with the result of [7], we have

Corollary 3. *Let* $\overline{M}^{n+p}$ *be an* $(n+p)$*-dimensional locally symmetric Bochner-Kaehler manifold and* M^n *an* $n(\geq 2)$*-dimensional compact Kaehler submanifold in* $\overline{M}^{n+p}$. *If the sectional curvature* $K > 0$ *for* M^n *and if* $p < n$, *then* M^n *is a totally geodesic* $\mathbb{C}P^n$ *in* $\overline{M}^{n+p}$.

Throughout this paper, all manifolds considered are smooth and connected unless otherwise stated.

2. Preliminaries

Let M^n be a Kaehler submanifold immersed in $\mathbb{C}P^{n+p}$. We will denote by J and $\langle\ ,\ \rangle$ the complex structure and the Fubini-Study metric of constant holomorphic sectional curvature 1 of $\mathbb{C}P^{n+p}$ as well as the induced complex structure and metric on M^n. Let σ, A and $\nabla^\perp$ be the second fundamental form, the Weingarten endomorphism and the normal connection for M^n in $\mathbb{C}P^{n+p}$, respectively. Then, we have the relations

$$\sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y), \tag{1}$$

$$A_{J\xi} = JA_\xi = -A_\xi J, \qquad \nabla^\perp_X J\xi = J\nabla^\perp_X \xi, \tag{2}$$

where X, Y are tangent to M^n and ξ is normal to M^n in $\mathbb{C}P^{n+p}$.

The Gauss equation of M^n in $\mathbb{C}P^{n+p}$ is

$$\begin{aligned} R(X, Y)Z = \tfrac{1}{4}(\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY \\ + 2\langle X, JY\rangle JZ) \\ + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y, \end{aligned} \tag{3}$$

where R is the Riemannian curvature tensor of M^n.

If u is a unit tangent vector to M^n, then, by (3), the holomorphic sectional curvature $H(u)$ of M^n determined by u can be expressed as

$$H(u) = 1 - 2\|\sigma(u, u)\|^2. \tag{4}$$

Let $\pi : UM \to M$ and UM_x be the unit tangent bundle over M^n and its fiber at $x \in M^n$, respectively. Then we define a function $f : UM \to \mathbf{R}$ by $f(u) = \|\sigma(u, u)\|^2$ for any $u \in UM_x$ and $x \in M^n$. If M^n is compact, then so is UM. Thus, the function f attains its maximum at some vector $v \in UM_x$ for some $x \in M^n$. Then from [2] we have

$$f(v)(1 - 4f(v)) \leq 0, \tag{5}$$

$$A_{\sigma(v,v)}v = f(v) \cdot v. \tag{6}$$

Fix v in UM_x. For any $u \in UM_x$, let $\gamma_u(t)$ be the geodesic in M^n determined by the initial conditions $\gamma_u(0) = x$ and $\gamma_x'(0) = u$. The parallel translation of v along $\gamma_u(t)$ yields a vector field $V_u(t)$. Then the function $f_u(t) = f(V_u(t))$ attains a maximum at $t = 0$, which implies that

$$\frac{d^2}{dt^2}f_u(0) + \frac{d^2}{dt^2}f_{Ju}(0) \leq 0$$

for all $u \in UM_x$. By direct computations we obtain from the above (cf. [3], the formula (13))

$$2f(v)\langle R(u, Ju)Jv, v\rangle - \tfrac{1}{2}f(v) - 2\|A_{\sigma(v,v)}u\|^2 \leq 0 \tag{7}$$

for all $u \in UM_x$.

Let $K(u, v)$ denote the sectional curvature of M^n at x for the plane spanned by $u, v \in UM_x$. The Bianchi identity shows that

$$\langle R(u, Ju)Jv, v\rangle = K(u, v) + K(u, Jv),$$

which together with (7) yields

$$f(v)\left\{K(u, v) + K(u, Jv) - \tfrac{1}{4}\right\} - \|A_{\sigma(v,v)}u\|^2 \leq 0 \tag{8}$$

for all $u \in UM_x$ such that $\langle u, v\rangle = \langle u, Jv\rangle = 0$.

On the other hand, from (3) it follows that

$$K(u, v) + K(u, Jv) = \tfrac{1}{2} - 2\|\sigma(u, v)\|^2 \leq \tfrac{1}{2}. \tag{9}$$

Combining (8) with (9), we have

$$f(v)(1 - 8\|\sigma(u, v)\|^2) - 4\|A_{\sigma(v,v)}u\|^2 \leq 0 \tag{10}$$

for all $u \in UM_x$ with $\langle u, v\rangle = \langle u, Jv\rangle = 0$.

Finally, it is remarkable that a complete classification for compact Kaehler submanifolds in $\mathbb{C}P^{n+p}$ with parallel second fundamental form has been given by H. Nakagawa and R. Takagi ([8]); see also M. Takeuchi ([9]).

3. Proof of the Theorem

To prove the Theorem, we need first to establish some lemmas.

Let M^n be an $n(\geq 2)$-dimensional compact Kaehler submanifold immersed in $\mathbb{C}P^{n+p}$. As is said in §2, the function $f : UM \to \mathbf{R}$ attains its maximum at the vector $v \in UM_x$ for the point $x \in M^n$.

Lemma 1. *Suppose that M^n is not totally geodesic in $\mathbb{C}P^{n+p}$. Then*

$$f(v) \geq \tfrac{1}{4},$$

where the equality holds if and only if $\widetilde{\nabla}\sigma = 0$, i.e., M^n has parallel second fundamental form in $\mathbb{C}P^{n+p}$.

Proof. The hypothesis of the lemma implies that $f(v) > 0$. Thus, the inequality follows directly from (5). Furthermore, by (4), we see that the equality holds if and only if

$$H(u) \geq 1 - 2\|\sigma(v, v)\|^2 = \tfrac{1}{2}$$

for all $u \in UM$. Now, the conclusion follows from [4]. □

Lemma 2. *Suppose that $\widetilde{\nabla}\sigma \neq 0$. Set $\xi = \sigma(v, v)/\|\sigma(v, v)\|$. If M^n has nonnegative sectional curvature, then*

$$\|A_\xi u\|^2 < \tfrac{1}{4} \tag{11}$$

for any unit eigenvector u of $A_{\sigma(v,v)}$ with $\langle u, v\rangle = \langle u, Jv\rangle = 0$.

Proof. Since the sectional curvature $K \geq 0$, it follows from (9) that

$$\tfrac{1}{2} + K(u, v) - K(u, Jv) \geq 2K(u, v) \geq 0,$$
$$\tfrac{1}{2} - K(u, v) + K(u, Jv) \geq 2K(u, Jv) \geq 0,$$

which imply that

$$\{K(u, v) - K(u, Jv)\}^2 \leq \tfrac{1}{4} \tag{12}$$

for all $u \in UM_x$ with $\langle u, v\rangle = \langle u, Jv\rangle = 0$.

From (6) we see that v and Jv are eigenvectors of $A_{\sigma(v,v)}$. Since $n \geq 2$, one can always choose a unit eigenvector u of $A_{\sigma(v,v)}$ such that $\langle u, v\rangle = \langle u, Jv\rangle = 0$. Thus, from (3) we have (cf. [3])

$$A_{\sigma(v,v)}u = \tfrac{1}{2}\{K(u, v) - K(u, Jv)\}u,$$

which together with (12) yields that

$$\|A_{\sigma(v,v)}u\|^2 \leq \tfrac{1}{16}. \tag{13}$$

Since $\widetilde{\nabla}\sigma \neq 0$, by Lemma 1, from (13) we obtain

$$\|A_{\sigma(v,v)}u\|^2 < \tfrac{1}{4}\|\sigma(v, v)\|^2,$$

which implies (11). □

Lemma 3. *Under the same hypothesis as in Lemma 2, we have*

$$\|\sigma(u, v)\|^2 > 0 \tag{14}$$

for any $u \in UM_x$ with $\langle u, v\rangle = \langle u, Jv\rangle = 0$.

Proof. Since v and Jv are eigenvectors of $A_{\sigma(v,v)}$ as well as A_ξ, we can choose unit vectors $e_3, \dots, e_{2n}$ in UM_x such that $\{v, Jv, e_3, \dots, e_{2n}\}$ is an orthonormal basis in T_xM and $A_\xi e_k = \lambda_k e_k$ for $k = 3, \dots, 2n$. By Lemma 2, we then have

$$\lambda_k^2 < \tfrac{1}{4} \quad \text{for } k = 3, \dots, 2n. \tag{15}$$

Let $u \in UM_x$ with $\langle u, v\rangle = \langle u, Jv\rangle = 0$. Then u can be expressed as

$$u = \sum_{k=3}^{2n} a_k e_k, \qquad \sum_{k=3}^{2n} a_k^2 = 1.$$

Thus, by (15), we have

$$\begin{aligned} \|A_\xi u\|^2 &= \left\|\sum_{k=3}^{2n} a_k (A_\xi e_k)\right\|^2 = \sum_{k=3}^{2n} a_k^2 \lambda_k^2 \\ &< \frac{1}{4}\sum_{k=3}^{2n} a_k^2 = \frac{1}{4}. \end{aligned} \tag{16}$$

On the other hand, since $f(v) \neq 0$, (10) is then equivalent to

$$1 - 8\|\sigma(u, v)\|^2 - 4\|A_\xi u\|^2 \leq 0.$$

From this and (16) the lemma follows. □

Proof of Theorem. By the result of [8], it is enough to prove that M^n has parallel second fundamental form, i.e., $\tilde{\nabla}\sigma = 0$.

We use reductio ad absurdum. Suppose that $\tilde{\nabla} \neq 0$, which implies that $f(v) > 0$. Let $e_1 = v$, $e_2 = Jv$, $e_3, \dots, e_{2n}$ be unit eigenvectors of $A_{\sigma(v,v)}$ so that $\{e_i\}_{1 \leq i \leq 2n}$ is an orthonormal basis in $T_x M$.

Since $p < n$, the normal vectors $\sigma(v, e_i)$, $i = 1, \dots, 2n$, are linearly dependent; i.e., there exist scalars a_i with at least one nonzero scalar such that

$$\sum_{i=1}^{2n} a_i \sigma(v, e_i) = 0. \tag{17}$$

On the other hand, by (6), we have

$$\langle \sigma(v, v), \sigma(v, e_i)\rangle = \langle A_{\sigma(v,v)} v, e_i\rangle = f(v)\langle e_1, e_i\rangle.$$

From this and (17) it follows that $a_1 = 0$. In the same way, by using a similar argument for Jv and (17), we can easily see that $a_2 = 0$. Thus, (17) becomes indeed

$$\sigma\left(v, \sum_{k=3}^{2n} a_k e_k\right) = 0, \tag{18}$$

where at least one of $\{a_k\}_{3 \leq k \leq 2n}$ is not zero.

We now choose the unit vector

$$u = \left(\sum_{k=3}^{2n} a_k e_k\right) \Big/ \left(\sum_{k=3}^{2n} a_k^2\right)^{1/2}$$

which satisfies $\langle u, v\rangle = \langle u, Jv\rangle = 0$, obviously. Then, by (18), we have

$$\sigma(v, u) = 0,$$

which contradicts Lemma 3. Hence, the theorem is proved completely. □

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