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AMENABILITY AND UNIFORMLY CONTINUOUS FUNCTIONALS ON THE ALGEBRAS $A_p(G)$ FOR DISCRETE GROUPS

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ABSTRACT. In this paper, it is shown that for every discrete group G and $1 < p < \infty$, $A_p(G) \cdot PM_p(G)$ is closed in $PM_p(G)$ if and only if G is amenable.

Let G be a locally compact group, $1 < p < \infty$ and $A_p(G)$ be the Herz-Figà-Talamanca algebra consists of functions f on G which can be represented as

$$f = \sum u_n * v_n^\vee$$

where $u_n \in L^q(G)$, $v_n \in L^p(G)$ $(p^{-1} + q^{-1} = 1)$ with $\sum \|u_n\|_q \|v_n\|_p < \infty$, with norm as the infimum of the last expression over all such representations of f.

The space of multipliers of $A_p(G)$ is denoted by $MA_p(G)$, i.e., $MA_p(G) = \{u \in C(G) \colon uA_p(G) \subseteq A_p(G)\}$, where $C(G)$ is the space of bounded continuous functions on G. The norm of u in $MA_p(G)$ is

$$\|u\|_{MA_p} = \sup\{\|uv\|_{A_p} \colon v \in A_p(G),\ \|v\|_{A_p} \le 1\}.$$

Let $CONV_p(G)$ denote the bounded linear operators on $L^p(G)$ which commute with translations from the right, equipped with operator norm. Let $PF_p(G)$ and $PM_p(G)$ be the norm closure and weak-star closure of $L^1(G)$ in $CONV_p(G)$. It turns out that $PM_p(G)$ is just the Banach space dual of $A_p(G)$. It should be mentioned that $A_2(G) = A(G)$ is the Fourier algebra of G and $PM_2(G) = VN(G)$ is the von Neumann algebra of G; see [4]. There is a natural action $MA_p(G)$ on $PM_p(G)$: given $u \in MA_p(G)$ and $f \in PM_p(G)$, define $u \cdot f$ as

$$\langle u \cdot f, v\rangle = \langle f, uv\rangle$$

for all $v \in A_p(G)$. When G is discrete, every element of $PM_p(G)$ can be identified with a left convolution operator by a function of $l^p(G)$. So in this situation, the action is simply the pointwise multiplication of functions.

The norm closure of $A_p(G) \cdot PM_p(G)$ in $PM_p(G)$ is denoted by $UC_p(\widehat{G})$ and it is called p-uniformly continuous functionals of $A_p(G)$.

The reader is referred to [9, 10] for more information of these spaces.

As stated in [9], the space $UC_P(\widehat{G})$ coincides with the PM_p-norm closure of the set of $T \in PM_p(G)$ with compact support (for the definition of support see [10]).

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It was observed by Granirer [8] that if G is an amenable locally compact group, then $UC_p(\widehat{G}) = A_p(G) \cdot PM_p(G)$, or, in other words, $A_p(G) \cdot PM_p(G)$ is a closed subspace of $PM_p(G)$. More results concerning this topic were obtained for $p = 2$: Figà-Talamanca showed that for the free group G on 2 generators, $A_2(G) \cdot PM_2(G)$ is not closed in $PM_2(G)$ (see [7]); this fact is even true for every nonamenable group as it was shown by Chou for the discrete case and by Lau and Losert for the general case (see [12]).

The purpose of this paper is to extend Chou's result to the case of $1 < p < \infty$. To this end, we will first describe a predual of the Banach space $MA_p(G)$.

Lemma. *Let G be a discrete group. Then the unit ball of $MA_p(G)$ is $\sigma(l^\infty, l^1)$-closed.*

Proof. Let B be the unit ball of $MA_p(G)$. Let $\{\phi_\alpha\}$ be a net of B such that $\phi_\alpha \to \phi$ for some $\phi \in l^\infty(G)$ in $\sigma(l^\infty, l^1)$-topology, i.e., $\phi_\alpha(x) \to \phi(x)$ for every $x \in G$.

For $v \in A_p(G)$ with finite support, we have

$$\|\phi_\alpha v - \phi v\|_{A_p} \le \|\phi_\alpha v - \phi v\|_p \to 0.$$

Since $\phi_\alpha \in B$, $\|\phi_\alpha v\|_{A_p} \le \|v\|_{A_p}$, hence

$$\|\phi v\|_{A_p} \le \|v\|_{A_p}.$$

Now for arbitrary $v \in A_p(G)$, notice that the set of functions with finite support is dense in $A_p(G)$, so we can find a sequence $\{v_n\}$ of $A_p(G)$, each v_n is of finite support, and $\|v_n - v\|_{A_p} \to 0$. Since $v_n - v_m$ is of finite support, we have

$$\|\phi v_n - \phi v_m\|_{A_p} \le \|v_n - v_m\|_{A_p}$$

for all m, n. So $\{\phi v_n\}$ is a Cauchy sequence of $A_p(G)$. Therefore, there exists a function $u \in A_p(G)$ such that $\|\phi v_n - u\|_{A_p} \to 0$. In particular,

$$\phi(x) v_n(x) \to u(x)$$

for every $x \in G$. On the other hand, we already have

$$\phi(x) v_n(x) \to \phi(x) v(x)$$

for every $x \in G$. It follows that $\phi v = u \in A_p(G)$, and hence $\phi \in MA_p(G)$.

It is easy to see that $\|\phi v\|_{A_p} \le \|v\|_{A_p}$, thus $\phi \in B$. □

Let G be a locally compact group, dx be a fixed left Haar measure on G. Let Q_p be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_{Q_p} = \sup\left\{\left|\int_G f(x)\phi(x)dx\right| : \phi \in MA_p(G), \|\phi\|_{MA_p} \le 1\right\}$$

for $f \in L^1(G)$.

In [4], De Cannère and Haagerup showed that $MA_2(G)$ is the dual of the Banach space Q_2. The following proposition shows that a similar result holds true for any $1 < p < \infty$ provided G is discrete.

Proposition. *Let G be a discrete group. Then the Banach space dual of Q_p is $MA_p(G)$.*

Proof. The proof is similar to that of Proposition 1.10 of [4]. We give it here for the sake of completeness.

We will establish the following correspondence: for each bounded linear functional to α on Q_p, there exists a $\phi \in MA_p(G)$ such that

$$\langle \alpha , f \rangle = \sum_{x \in G} f(x)\phi(x)$$

for all $f \in l^1(G)$ and

$$\|\alpha\| = \|\phi\|_{MA_p}.$$

It is clear that for $\phi \in MA_p(G)$, the bounded linear functional α_ϕ in $l^1(G)$ defined by

$$\langle \alpha_\phi , f \rangle = \sum_{x \in G} f(x)\phi(x)$$

can be extended to a bounded linear functional on Q_p (still denoted by α_ϕ) and $\|\alpha_\phi\| \leq \|\phi\|_{MA_p}$.

Conversely, let α be a bounded linear functional on Q_p with $\|\alpha\| = 1$. First we show that the restriction of α on $l^1(G)$ is bounded. In fact, for $\phi \in MA_p(G)$, since $\|\phi\|_\infty \leq \|\phi\|_{MA_p}$, it follows that for $f \in l^1(G)$, $\|f\|_{Q_p} \leq \|f\|_1$ by the definition of $\|f\|_{Q_p}$.

Therefore, for $f \in l^1(G)$,

$$|\langle \alpha , f \rangle| \leq \|\alpha\| \, \|f\|_{Q_p} \leq \|f\|_1.$$

So $\alpha|_{l^1(G)}$ is bounded, and hence there exists $\psi \in l^\infty(G)$ such that

$$\langle \alpha , f \rangle = \sum_{x \in G} f(x)\psi(x)$$

for every $f \in l^1(G)$.

Notice that the unit ball B of $MA_p(G)$ is a convex balanced subset of the locally convex space $(l^\infty(G), \sigma(l^\infty, l^1))$. So by the bipolar theorem, the $\sigma(l^\infty, l^1)$-closure of B is ${}^0B^0$, the bipolar of B. Hence by the lemma, ${}^0B^0 = B$. From the definition of $\|\cdot\|_{Q_p}$, $B^0 = \{h \in l^1(G) \colon \|h\|_{Q_p} \leq 1\}$. Since for any $h \in B^0$, $|\langle \psi , h \rangle| \leq 1$, so $\psi \in {}^0B^0$, and hence $\psi \in B \subseteq MA_p(G)$.

The proof is complete. □

Now we are ready for our main result.

Theorem. *Let G be a discrete group. Then $A_p(G)\cdot PM_p(G)$ is closed in $PM_p(G)$ if and only if G is amenable.*

Proof. We need to show the only if part.

For $f \in l^1(G)$, we have

$$\begin{aligned} \|f\|_{Q_p} &= \sup\left\{\left|\sum_{x \in G} f(x)\phi(x)\right| : \phi \in MA_p(G), \|\phi\|_{MA_p} \leq 1\right\} \\ &\geq \sup\left\{\left|\sum_{x \in G} f(x)v(x)\right| : v \in A_p(G), \|v\|_{A_p} \leq 1\right\} \\ &= \|f\|_{PM_p}. \end{aligned}$$

Since $PF_p(G)$ is the $\|\cdot\|_{PM_p}$-closure of $l^1(G)$, we conclude that $Q_p \subseteq PF_p(G)$.

On the other hand, for every $v \in A_p(G)$ with finite support and for every $f \in PM_p(G)$, since $v \cdot f \in l^1(G)$, we get

$$\begin{aligned}\|v \cdot f\|_{Q_p} &= \sup\{|\langle v \cdot f, \phi\rangle| : \phi \in MA_p(G), \|\phi\|_{MA_p} \le 1\} \\ &= \sup\{|\langle f, \phi v\rangle| : \phi \in MA_p(G), \|\phi\|_{MA_p} \le 1\} \\ &\le \sup\{|\langle f, w\rangle| : w \in A_p(G), \|w\|_{A_p} \le \|v\|_{A_p}\} \\ &\le \|f\|_{PM_p}\|v\|_{A_p}.\end{aligned}$$

By passing through the limit, we can see that for any $v \in A_p(G)$ and any $f \in PM_p(G)$, $v \cdot f \in Q_p$ and $\|v \cdot f\|_{Q_p} \le \|f\|_{PM_p}\|v\|_{A_p}$. Hence $A_p(G) \cdot PM_p(G) \subseteq Q_p$.

It was shown by Granirer [9] that $UC_p(\widehat{G}) = PF_p(G)$ if G is discrete. Since $A_p(G) \cdot PM_p(G)$ is closed, the equality $PF_p(G) = A_p(G) \cdot PM_p(G)$ holds. Therefore, $Q_p = PF_p(G)$. Since $\|\cdot\|_{Q_p} \ge \|\cdot\|_{PM_p}$, by the open mapping theorem, $\|\cdot\|_{Q_p} \le C\|\cdot\|_{PM_p}$ for some constant $C > 0$.

Note that for $\chi_G \in MA_p(G)$ with $\|\chi_G\|_{MA_p} = 1$, we have

$$\|f\|_{Q_p} \ge \left|\sum_{x \in G} f(x)\chi_G(x)\right| = \|f\|_1$$

for every $f \in l^1(G)$ with $f \ge 0$. Hence $\|f\|_1 \le C\|f\|_{PM_p}$. Applying the estimate to the n-fold convolution of f with itself we get

$$\|f\|_1^n \le C\|f\|_{PM_p}^n,$$

so

$$\|f\|_1 \le \|f\|_{PM_p}.$$

This shows that $\|f\|_1 = \|f\|_{PM_p}$ for every $f \in l^1(G)$ with $f \ge 0$. By a result of Leptin [13, Theorem 1], G is amenable. □

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