AMENABILITY AND UNIFORMLY CONTINUOUS FUNCTIONALS
ON THE ALGEBRAS $A_p(G)$ FOR DISCRETE GROUPS

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Abstract. In this paper, it is shown that for every discrete group $G$ and $1 < p < \infty$, $A_p(G) \cdot PM_p(G)$ is closed in $PM_p(G)$ if and only if $G$ is amenable.

Let $G$ be a locally compact group, $1 < p < \infty$ and $A_p(G)$ be the Herz-Figà-Talamanca algebra consists of functions $f$ on $G$ which can be represented as

$$f = \sum u_n * v_n^*$$

where $u_n \in L^q(G)$, $v_n \in L^p(G)$ $(p^{-1} + q^{-1} = 1)$ with $\sum \|u_n\|_q \|v_n\|_p < \infty$, with norm as the infimum of the last expression over all such representations of $f$.

The space of multipliers of $A_p(G)$ is denoted by $MA_p(G)$, i.e., $MA_p(G) = \{u \in C(G) : uA_p(G) \subseteq A_p(G)\}$, where $C(G)$ is the space of bounded continuous functions on $G$. The norm of $u$ in $MA_p(G)$ is

$$\|u\|_{MA_p} = \sup\{\|uv\|_{A_p} : v \in A_p(G), \|v\|_{A_p} \leq 1\}.$$

Let $CONV_p(G)$ denote the bounded linear operators on $L^p(G)$ which commute with translations from the right, equipped with operator norm. Let $PF_p(G)$ and $PM_p(G)$ be the norm closure and weak-star closure of $L^1(G)$ in $CONV_p(G)$. It turns out that $PM_p(G)$ is just the Banach space dual of $A_p(G)$. It should be mentioned that $A_2(G) = A(G)$ is the Fourier algebra of $G$ and $PM_2(G) = VN(G)$ is the von Neumann algebra of $G$; see [4]. There is a natural action $MA_p(G)$ on $PM_p(G)$: given $u \in MA_p(G)$ and $f \in PM_p(G)$, define $u \cdot f$ as

$$\langle u \cdot f, v \rangle = \langle f, uv \rangle$$

for all $v \in A_p(G)$. When $G$ is discrete, every element of $PM_p(G)$ can be identified with a left convolution operator by a function of $l^p(G)$. So in this situation, the action is simply the pointwise multiplication of functions.

The norm closure of $A_p(G) \cdot PM_p(G)$ in $PM_p(G)$ is denoted by $UC_p(\hat{G})$ and it is called $p$-uniformly continuous functionals of $A_p(G)$.

The reader is referred to [9, 10] for more information of these spaces.

As stated in [9], the space $UC_p(\hat{G})$ coincides with the $PM_p$-norm closure of the set of $T \in PM_p(G)$ with compact support (for the definition of support see [10]).

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It was observed by Granirer [8] that if $G$ is an amenable locally compact
group, then $UCP(G) = A_p(G) \cdot PM_p(G)$, or, in other words, $A_p(G) \cdot PM_p(G)$ is
a closed subspace of $PM_p(G)$. More results concerning this topic were obtained
for $p = 2$: Figà-Talamanca showed that for the free group $G$ on 2 generators,$A_2(G) \cdot PM_2(G)$ is not closed in $PM_2(G)$ (see [7]); this fact is even true for
every nonamenable group as it was shown by Chou for the discrete case and by
Lau and Losert for the general case (see [12]).

The purpose of this paper is to extend Chou’s result to the case of $1 < p < \infty$.
To this end, we will first describe a predual of the Banach space $MA_p(G)$.

**Lemma.** Let $G$ be a discrete group. Then the unit ball of $MA_p(G)$ is $\sigma(l^\infty, l^1)$-
closed.

**Proof.** Let $B$ be the unit ball of $MA_p(G)$. Let $\{\phi_a\}$ be a net of $B$ such that
$\phi_a \to \phi$ for some $\phi \in l^\infty(G)$ in $\sigma(l^\infty, l^1)$-topology, i.e., $\phi_a(x) \to \phi(x)$ for
every $x \in G$.

For $v \in A_p(G)$ with finite support, we have

$$\|\phi_a v - \phi v\|_A \leq \|\phi_a v - \phi v\|_p \to 0.$$ 

Since $\phi_a \in B$, $\|\phi_a v\|_A \leq \|v\|_A$, hence

$$\|\phi v\|_A \leq \|v\|_A.$$ 

Now for arbitrary $v \in A_p(G)$, notice that the set of functions with finite
support is dense in $A_p(G)$, so we can find a sequence $\{v_n\}$ of $A_p(G)$, each $v_n$
is of finite support, and $\|v_n - v\|_A \to 0$. Since $v_n - v_m$ is of finite support, we
have

$$\|\phi v_n - \phi v_m\|_A \leq \|v_n - v_m\|_A,$$

for all $m, n$. So $\{\phi v_n\}$ is a Cauchy sequence of $A_p(G)$. Therefore, there exists
a function $u \in A_p(G)$ such that $\|\phi v_n - u\|_A \to 0$. In particular,

$$\phi(v_n(x)) \to u(x)$$

for every $x \in G$. On the other hand, we already have

$$\phi(x)v_n(x) \to \phi(x)u(x)$$

for every $x \in G$. It follows that $\phi v = u \in A_p(G)$, and hence $\phi \in MA_p(G)$.

It is easy to see that $\|\phi v\|_A \leq \|v\|_A$, thus $\phi \in B$. $\square$

Let $G$ be a locally compact group, $dx$ be a fixed left Haar measure on $G$.
Let $Q_p$ be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_{Q_p} = \sup \left\{ \left| \int_G f(x)\phi(x)dx \right| : \phi \in MA_p(G), \|\phi\|_{MA_p} \leq 1 \right\}$$

for $f \in L^1(G)$.

In [4], De Cannère and Haagerup showed that $MA_2(G)$ is the dual of the
Banach space $Q_2$. The following proposition shows that a similar result holds
true for any $1 < p < \infty$ provided $G$ is discrete.

**Proposition.** Let $G$ be a discrete group. Then the Banach space dual of $Q_p$ is
$MA_p(G)$.

**Proof.** The proof is similar to that of Proposition 1.10 of [4]. We give it here
for the sake of completeness.
We will establish the following correspondence: for each bounded linear functional to $\alpha$ on $Q_p$, there exists a $\phi \in MAP(G)$ such that
\[
\langle \alpha, f \rangle = \sum_{x \in G} f(x)\phi(x)
\]
for all $f \in l^1(G)$ and
\[
\|\alpha\| = \|\phi\|_{MAP}.
\]

It is clear that for $\phi \in MAP(G)$, the bounded linear functional $\alpha_\phi$ in $l^1(G)$ defined by
\[
\langle \alpha_\phi, f \rangle = \sum_{x \in G} f(x)\phi(x)
\]
can be extended to a bounded linear functional on $Q_p$ (still denoted by $\alpha_\phi$) and $\|\alpha_\phi\| \leq \|\phi\|_{MAP}$.

Conversely, let $\alpha$ be a bounded linear functional on $Q_p$ with $\|\alpha\| = 1$. First we show that the restriction of $\alpha$ on $l^1(G)$ is bounded. In fact, for $\phi \in MAP(G)$, since $\|\phi\|_{\infty} \leq \|\phi\|_{MAP}$, it follows that for $f \in l^1(G)$, $\|f\|_{Q_p} \leq \|f\|_1$ by the definition of $\|f\|_{Q_p}$.

Therefore, for $f \in l^1(G)$,
\[
\langle \alpha, f \rangle \leq \|\alpha\| \|f\|_{Q_p} \leq \|f\|_1.
\]
So $\alpha|l^1(G)$ is bounded, and hence there exists $\psi \in l^\infty(G)$ such that
\[
\langle \alpha, f \rangle = \sum_{x \in G} f(x)\psi(x)
\]
for every $f \in l^1(G)$.

Notice that the unit ball $B$ of $MAP(G)$ is a convex balanced subset of the locally convex space $(l^\infty(G), \sigma(l^\infty, l^1))$. So by the bipolar theorem, the $\sigma(l^\infty, l^1)$-closure of $B$ is $0B^0_1$, the bipolar of $B$. Hence by the lemma, $0B^0_1 = B$. From the definition of $\|\cdot\|_{Q_p}$, $B^0_1 = \{h \in l^1(G) : \|h\|_{Q_p} \leq 1\}$. Since for any $h \in B^0_1$, $|\langle \psi, h \rangle| \leq 1$, so $\psi \in 0B^0_1$, and hence $\psi \in B \subseteq MAP(G)$.

The proof is complete. $\square$

Now we are ready for our main result.

**Theorem.** Let $G$ be a discrete group. Then $AP(G) \cdot PM_p(G)$ is closed in $PM_p(G)$ if and only if $G$ is amenable.

**Proof.** We need to show the only if part.

For $f \in l^1(G)$, we have
\[
\|f\|_{Q_p} = \sup \left\{ \sum_{x \in G} f(x)\phi(x) : \phi \in MAP(G), \|\phi\|_{MAP} \leq 1 \right\}
\geq \sup \left\{ \sum_{x \in G} f(x)v(x) : v \in AP(G), \|v\|_{AP} \leq 1 \right\}
= \|f\|_{PM_p}.
\]

Since $PF_p(G)$ is the $\|\cdot\|_{PM_p}$-closure of $l^1(G)$, we conclude that $Q_p \subseteq PF_p(G)$. 

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On the other hand, for every $v \in A_p(G)$ with finite support and for every $f \in PM_p(G)$, since $v \cdot f \in l^1(G)$, we get

$$\|v \cdot f\|_{Q_p} = \sup\{|(v \cdot f, \phi)|: \phi \in MA_p(G), \|\phi\|_{MA_p} \leq 1\}$$

$$= \sup\{|(f, \phi v)|: \phi \in MA_p(G), \|\phi\|_{MA_p} \leq 1\}$$

$$\leq \sup\{|(f, w)|: w \in A_p(G), \|w\|_{A_p} \leq \|v\|_{A_p}\}$$

$$\leq \|f\|_{PM_p} \|v\|_{A_p}.$$

By passing through the limit, we can see that for any $v \in A_p(G)$ and any $f \in PM_p(G)$, $v \cdot f \in Q_p$ and $\|v \cdot f\|_{Q_p} \leq \|f\|_{PM_p} \|v\|_{A_p}$. Hence $A_p(G) \cdot PM_p(G) \subseteq Q_p$.

It was shown by Granirer [9] that $UC_p(\hat{G}) = PF_p(G)$ if $G$ is discrete. Since $A_p(G) \cdot PM_p(G)$ is closed, the equality $PF_p(G) = A_p(G) \cdot PM_p(G)$ holds. Therefore, $Q_p = PF_p(G)$. Since $\|\cdot\|_{Q_p} \geq \|\cdot\|_{PM_p}$, by the open mapping theorem, $\|\cdot\|_{Q_p} \leq C \|\cdot\|_{PM_p}$ for some constant $C > 0$.

Note that for $\chi_G \in MA_p(G)$ with $\|\chi_G\|_{MA_p} = 1$, we have

$$\|f\|_{Q_p} \geq \left| \sum_{x \in G} f(x) \chi_G(x) \right| = \|f\|_1$$

for every $f \in l^1(G)$ with $f \geq 0$. Hence $\|f\|_1 \leq C \|f\|_{PM_p}$. Applying the estimate to the $n$-fold convolution of $f$ with itself we get

$$\|f\|_1^n \leq C \|f\|_{PM_p}^n,$$

so

$$\|f\|_1 \leq \|f\|_{PM_p}.$$

This shows that $\|f\|_1 = \|f\|_{PM_p}$ for every $f \in l^1(G)$ with $f \geq 0$. By a result of Leptin [13, Theorem 1], $G$ is amenable. □

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