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SOLUTION OF THE BAIRE ORDER PROBLEM OF MAULDIN

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ABSTRACT. Let X be an uncountable Polish space, and let I be a proper σ-ideal of subsets of X such that $\{x\} \in I$ for each $x \in X$. Denote by $B_\alpha(I)$, $\alpha \leq \omega_1$, the Baire system generated by the family of functions $f : X \to \mathbb{R}$ continuous I almost everywhere. We prove that if $r(I) = \min\{\alpha \leq \omega_1 : B_{\alpha+1}(I) = B_\alpha(I)\}$, then either $r(I) = 1$ or $r(I) = \omega_1$. This answers the problem raised by R. D. Mauldin in 1973.

1. INTRODUCTION

Let X be an uncountable separable and complete metric space (briefly called *Polish*), and let I be a σ-ideal of subsets of X such that $X \notin I$. Denote by C_I the family of all functions $f : X \to \mathbb{R}$ whose sets of points of discontinuity are in I. Then put $B_0(I) = C_I$ and for each ordinal $\alpha > 0$ define $B_\alpha(I)$ as the family of all pointwise limits of sequences of functions from $\bigcup_{\gamma<\alpha} B_\gamma(I)$.

It is easy to check that the *Baire system* $B_\alpha(I)$, $\alpha \leq \omega_1$, has the following properties:

- $B_{\omega_1}(I)$ is closed under pointwise limits, i.e. $B_{\omega_1+1}(I) = B_{\omega_1}(I)$,
- for $I = \{\emptyset\}$ we have the classical Baire system (denoted by B_α, $\alpha \leq \omega_1$).

Now we define $r(I) = \min\{\alpha \leq \omega_1 : B_{\alpha+1}(I) = B_\alpha(I)\}$ which is called the *Baire order* of C_I.

The following results are known:

(1) If $I = \{\emptyset\}$, then $r(I) = \omega_1$ (see [L]).

(2) If I is the σ-ideal of the first category sets in X, then $r(I) = 1$ and $B_1(I)$ consists of all functions with the Baire property (see [K1]).

(3) If I is the σ-ideal of sets of Lebesgue measure zero in $[0, 1]$, then $r(I) = \omega_1$ (see [M2]; for some generalizations compare [M3], [B1]).

In [M2] Mauldin posed the following problem: If $0 < \alpha < \omega_1$, is there a σ-ideal I_α of the first category subsets of $[0, 1]$ which contains all F_σ sets of Lebesgue measure 0 such that the family of all functions which are continuous except for a set in this σ-ideal I_α has Baire order α?

Note that in the above question, σ-ideals are required to contain all singletons $\{x\}$; a σ-ideal which has that property is called *uniform*. In the main theorem

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we will show that, for each uniform σ-ideal I of subsets of X, we have either $r(I) = 1$ or $r(I) = \omega_1$. It solves Mauldin's problem in the negative. Observe that always $r(I) > 0$ since the characteristic function of a countable set dense in X belongs to $B_1(I) \setminus B_0(I)$ (cf. [B2]).

Denote by $\mathscr{B}$ the family of all Borel subsets of X, and by Σ^0_α, Π^0_α (for $0 < \alpha < \omega_1$) the subclasses of $\mathscr{B}$ defined as in [Mo, 1 F]. In particular, Σ^0_2 is the family of all F_σ sets in X.

A σ-ideal I is called Σ^0_2 *supported* if each $A \in I$ is contained in some $B \in I \cap \Sigma^0_2$. For a σ-ideal I, we define

$$I^* = \{A \subset X : (\exists B \in I \cap \Sigma^0_2)(A \subset B)\}.$$

Obviously, I^* is a Σ^0_2 supported σ-ideal, and if I is a Σ^0_2 supported σ-ideal, then $I = I^*$. Since the set of discontinuity points of an arbitrary function is of type F_σ, we have $C_I = C_{I^*}$ for each σ-ideal I, and thus the Baire order problem may be restricted to Σ^0_2 supported σ-ideals.

2. Auxiliary facts

In this section we will assume that I is a uniform σ-ideal of subsets of X. If $\mathscr{F}$ is a family of subsets of X, then define $MGR(\mathscr{F})$ as the family of all subsets B of X such that for each $A \in \mathscr{F}$ the set $B \cap A$ is of the first category in A.

The following deep result plays a key role in the proof of our Main Theorem.

Proposition 1 [KS, Theorem 2]. *Let I be a Σ^0_2 supported σ-ideal. Then precisely one of the following posibilities holds*:

(i) $I = MGR(\mathscr{F})$ *for a countable family $\mathscr{F}$ of closed subsets of X (moreover, it may be assumed that $\mathscr{F} = \{F_\gamma : \gamma < \alpha\}$ where $\alpha < \omega_1$ and $F_\gamma \subset F_\beta$ for $\beta < \gamma < \alpha$, and $F_{\gamma+1}$ is nowhere dense in F_γ for $\gamma < \alpha$)*;

(ii) *there exists a homeomorphic embedding $\varphi : 2^\omega \times \omega^\omega \to X$ such that $\varphi[\{t\} \times \omega^\omega] \notin I$ for each $t \in 2^\omega$.*

Proposition 2. *If a σ-ideal I satisfies condition* (ii) *of Proposition* 1, *then I has the following property*:

(M) *There exists a Borel function $f : X \to X$ such that $f^{-1}[\{x\}] \notin I$ for each $x \in X$.*

Proof. Denote $B = \varphi[2^\omega \times \omega^\omega]$ and consider a continuous function $\psi = pr_1 \circ \varphi^{-1} : B \to 2^\omega$, where pr_1 is a projection onto the first factor. For each $t \in 2^\omega$ we have

$$\psi^{-1}[\{t\}] = \varphi[pr_1^{-1}[\{t\}]] = \varphi[\{t\} \times \omega^\omega] \notin I.$$

Since B is a Borel set (even of type G_δ, see [K, §35, III]), we can extend ψ to a Borel function $g : X \to 2^\omega$ and we get $g^{-1}[\{t\}] \notin I$ for each $t \in 2^\omega$. Let $h : 2^\omega \to X$ be a Borel isomorphism (see [K, §37, II]). Then $f = h \circ g : X \to X$ is a Borel function and for each $x \in X$ we have

$$f^{-1}[\{x\}] = g^{-1}[h^{-1}[\{x\}]] \notin I. \quad \square$$

Define $R(I) = \min\{\alpha \le \omega_1 : (\forall B \in \mathscr{B})(\exists A \in \Sigma^0_\alpha)(B \triangle A \in I)\}$ where $\Sigma^0_{\omega_1} = \mathscr{B}$ and $B \triangle A = (B \setminus A) \cup (A \setminus B)$. Observe that Σ^0_α can be replaced by Π^0_α in the above definition.

Proposition 3 [B3, Corollary 2.2]. *If a σ-ideal I has the property* (M), *then* $R(I) = \omega_1$.

Proposition 4 [M2, Theorem 3]. *For every σ-ideal I and each α, $0 < \alpha \leq \omega_1$, we have $f \in B_\alpha(I)$ if and only if there is $g \in B_\alpha$ such that $\{x \in X : f(x) \neq g(x)\} \in I^*$.*

Proposition 5 [B2]. *If I is a Σ_2^0 supported σ-ideal and $R(I) = \omega_1$, then $r(I) = \omega_1$.*

Proof. Suppose to the contrary that $r(I) = \alpha < \omega_1$, and consider an arbitrary set $E \in \mathscr{B}$. Then Proposition 4 yields that for each Borel function $f : X \to \mathbb{R}$ there is a function $g \in B_\alpha$ satisfying $\{x \in X : f(x) \neq g(x)\} \in I$. In particular, consider such a function g for $f = \chi_E$. Put $A = g^{-1}[\{1\}]$. Then $A \in \Pi_{\alpha+1}^0$ (cf. [K, §31, IX]) and $E \triangle A \subset \{x \in X : f(x) \neq g(x)\} \in I$. Hence $R(I) \leq \alpha + 1 < \omega_1$, which gives a contradiction. □

3. Main Theorem

Theorem. *If X is an uncountable Polish space and I is an uniform σ-ideal, then either $r(I) = 1$ or $r(I) = \omega_1$.*

Proof. According to our general remarks in Introduction, we may assume that I is Σ_2^0 supported and, by Proposition 1, we consider Cases (i) and (ii).

Case (i). Assume that $\mathscr{F} = \{F_\gamma : \gamma < \alpha\}$ for $\alpha < \omega_1$ and that the remaining conditions stated in (i) are fulfilled. Let S denote the set of all functions $f : X \to \mathbb{R}$ such that $f|F_\gamma$ has the Baire property for each $\gamma < \alpha$. Then $C_I \subset S$ and S is closed with respect to pointwise limits. Hence $B_\gamma(I) \subset S$ for each $\gamma < \omega_1$. Thus it suffices to show that $S \subset B_1(I)$. Let $f \in S$. By virtue of Kuratowski's result [K1] and Proposition 4, there exist functions $g_\gamma : F_\gamma \to \mathbb{R}$, $\gamma < \alpha$, of the Baire class 1, such that the set $\{x \in F_\gamma : (f|F_\gamma)(x) \neq g_\gamma(x)\}$ is of the first category in F_γ. Define $g : X \to \mathbb{R}$ by $g(x) = g_\gamma(x)$ for $x \in F_\gamma \setminus F_{\gamma+1}$, $\gamma < \alpha$, and

$$g(x) = 0 \quad \text{for } x \in (X \setminus F_0) \cup \bigcup_{\lambda<\alpha,\, \lambda \text{ limit}} \Big(\bigcap_{\gamma<\lambda} F_\gamma \setminus F_\lambda\Big).$$

Since

$$\{x \in X : f(x) \neq g(x)\} \cap F_\gamma \subset \{x \in F_\gamma \setminus F_{\gamma+1} : (f|F_\gamma)(x) \neq g_\gamma(x)\} \cup F_{\gamma+1},$$

therefore $\{x \in X : f(x) \neq g(x)\} \cap F_\gamma$ is of the first category in F_γ. Thus

$$\{x \in X : f(x) \neq g(x)\} \in MGR(\mathscr{F}) = I.$$

The function g is of the Baire class 1 since for each open subset $U \subset \mathbb{R}$ we have $g^{-1}[U] \in \Sigma_2^0$. Indeed,

$$g^{-1}[U] = \bigcup_{\gamma<\alpha} (g_\gamma^{-1}[U] \setminus F_{\gamma+1}) \quad \text{if } 0 \notin U,$$

and

$$(X \setminus F_0) \cup \bigcup_{\lambda<\alpha,\, \lambda \text{ limit}} \Big(\bigcap_{\gamma<\lambda} F_\gamma \setminus F_\lambda\Big)$$

must be added to the right side of the last equality, if $0 \in U$. Consequently, $f \in B_1(I)$ by Proposition 4.

Case (ii). Use Propositions 2, 3, and 5. □

Remark. At this moment we do not know which values between 1 and ω_1 can be achieved by $r(I)$ when I is not uniform. That question for non-uniform principal σ-ideals $I_A = \{E \subset X : E \subset X \setminus A\}$ where $A \subset X$ is uncountable was considered in [M3]. Note that $r(I_A) = \omega_1$ if A contains a perfect set (see [M3, Theorem 6]). So, it would be interesting to examine the case when A is uncountable and does not contain perfect sets.

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