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COHOMOLOGY RING OF THE ORBIT SPACE OF CERTAIN FREE Z_p-ACTIONS

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ABSTRACT. In this paper, we consider actions of $G = Z_p$ (with p an odd prime) on spaces X which are of cohomology type $(0, 0)$ (i.e., have the mod-p cohomology of the one-point union of an n-sphere, a $2n$-sphere and a a $3n$-sphere, n odd). If X is not totally non-homologous to zero in X_G we determine the fixed set, give examples of all possibilities for the fixed set and compute the cohomology ring structure of the orbit space in the case where G acts freely. In [4], we considered fixed sets for related spaces, when X is totally non-homologous to zero in X_G.

1. INTRODUCTION

Let X be a finite CW complex with cohomology groups satisfying:

$$H^j(X; Z) = \begin{cases} Z, & j = 0, n, 2n, 3n, \\ 0, & \text{otherwise}. \end{cases}$$

If u_i generates $H^{in}(X; Z)$, $i = 1, 2, 3$, we say that X has cohomology type (a, b) when $u_1^2 = au_2$ and $u_1u_2 = bu_3$ (terminology due to Toda [7]). Let $G = Z_p$ (p an odd prime) act on X. If $b \neq 0 \pmod p$, then either $X \simeq_p S^n \times S^{2n}$ or $X \simeq_p P^3(n)$ depending on whether $a = 0 \pmod p$ or $a \neq 0 \pmod p$. Here $X \simeq_p Y$ means that X and Y have isomorphic mod-p cohomology rings. When $b \neq 0 \pmod p$ the nature of the fixed set of G on X has been studied in detail ([5], [6], [8]). In [4] we considered the case $b = 0 \pmod p$, when X is totally non-homologous to zero in $X_G \pmod p$. The structure of the possible fixed sets was determined, and it was noted that when n is even, X is always totally non-homologous to zero. Here we settle the remaining case where X is not totally non-homologous to zero $\pmod p$, so that n is necessarily odd. Since n is odd, we must have $a = 0 \pmod p$ and X is of cohomology type $(0, 0)$. We obtain

Theorem 1. *Let $G = Z_p$, p an odd prime, act on a finite complex X of cohomology type $(0, 0) \pmod p$. If X is not totally non-homologous to zero in X_G, then the fixed point set $F \simeq_p S^q$, $-1 \leq q \leq 3n$, q odd. Moreover, all possibilities for q occur.*

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Examples of G acting freely on spaces X of type $(0, 0)$ were constructed in [4]. Here we compute the cohomology of the orbit space of a free G action on X obtaining

Theorem 2. *Let X be a space of cohomology type $(0, 0)$ with nonzero* mod-p *cohomology only in dimensions 0, n, $2n$, $3n$ (n odd). Suppose $G = Z_p$, p odd prime, acts freely on X. Then as graded commutative algebras,*

$$H^*(X/G; Z_p) = Z_p[x, y, z]/(x^2, z^2, zy^{n+1/2}, y^{3n+1/2})$$

where $\deg x = 1$, $\deg y = 2$, $\deg z = n$ *and* $y = \beta_p(x)$ *(β_p being the* mod-p *Bockstein).*

2. Preliminaries

We will recall here several facts about equivariant cohomology $H_G^*(X) = H^*(X_G)$ (see [2, Chapter 7] for more information).

First of all we will denote cohomology with Z_p coefficients simply by $H^*(X)$ and from now on Z_p coefficients are intended (unless explicitly indicated otherwise). If $G = Z_p$ acts on X, F denotes the fixed set. It is well known that if $H^*(X) = 0$ for $* > m$, then the inclusion $F_G \to X_G$ induces a cohomology isomorphism $H^*(X_G) \to H^*(F_G)$ for $* > m$. Recall that $X_G = EG \times_G X$ is the bundle over the classifying space BG with fibre X associated to the principal bundle $EG \to BG \cdot F_G = F \times BG$ is a subbundle.

If X is a finite G-CW-complex and G acts trivially on $H^*(X; Z)$ (integer coefficients), then for any k,

$$\sum_{i\geq 0} \operatorname{rk} H^{k+2i}(F) \leq \sum_{i\geq 0} \operatorname{rk} H^{k+2i}(X).$$

For example, if $H^*(X)$ vanishes in even degrees, so does $H^*(F)$.

We recall that if $\sum \operatorname{rk} H^*(X) < \infty$ (as in the present case), then X is totally non-homologous to zero in X_G (i.e., there is a cohomology extension of the fibre $X \to X_G$) with Z_p-coefficients iff $\sum \operatorname{rk} H^*(F) = \sum \operatorname{rk} H^*(X)$ iff G acts trivially on $H^*(X)$ and the Leray-Serre spectral sequence of $X^i \to X_G \xrightarrow{\pi} BG$ degenerates.

In computing the cohomology of the orbit space X/G, where G acts freely on X, we make use of the Leray-Serre spectral sequence of $X_G \to BG$ (in this case the map of orbit spaces $X_G \to X/G$ is a homotopy equivalence, so the cohomology of X_G obtained from this spectral sequence will be the cohomology of X/G). If $\pi_1(BG) = Z_p$ acts trivially on $H^*(X)$ (as it does in the present case), then the E_2 term is $E_2^{k,l} = H^k(BG) \otimes H^l(X)$. The product structure in the spectral sequence induces a product in the subalgebras $E_2^{*,0}$ and $E_2^{0,*}$ which coincides with the cup products. Also the edge homomorphisms,

$$H^k(BG) = E_2^{k,0} \twoheadrightarrow E_3^{k,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{k+1}^{k,0} = E_\infty^{k,0} \subseteq H^k(X_G),$$
$$H^l(X_G) \twoheadrightarrow E_\infty^{0,l} = E_{l+1}^{0,l} \subset \cdots \subset E_2^{0,l} = H^l(X)$$

are the homomorphisms $\pi^*: H^k(BG) \to H^k(X_G)$ and $i^*: H^l(X_G) \to H^l(X)$, respectively.

Finally recall that

$$H^*(BG) = Z_p[s, t]/(s^2) = \Lambda(s) \otimes Z_p[t]$$

where $\deg s = 1$, $\deg t = 2$ and $\beta_p(s) = t$ (β_p is the mod-p Bockstein associated to $Z_p \to Z_{p^2} \twoheadrightarrow Z_p$).

3. Proof of Theorem 1 and examples

To prove Theorem 1, suppose that X is not totally non-homologous to zero in X_G (mod p). Then we must have n odd (by [4]) and $\operatorname{rk} H^*(F) < \operatorname{rk} H^*(X) = 4$. Also $\chi(F) \equiv \chi(X) \equiv 0 \pmod p$. So $\chi(F) = 0$ or $\chi(F) = 3$ and $p = 3$. But $\chi(F) = 3$ requires $\sum \operatorname{rk} H^{2i}(F) > 2$ and, since G acts trivially on $H^*(X; Z)$ (integer coefficients), we must have

$$\sum \operatorname{rk} H^{2i}(F) \le \sum \operatorname{rk} H^{2i}(X) = 2.$$

Therefore $\chi(F) = 0$ and $F \simeq_p S^q$ for q odd and $-1 \le q \le 3n$. □

We now will give examples to show that all the possibilities for q are actually realised. The case of $q = -1$ (i.e., $F = \varnothing$) is [4]. So assume that q is odd $1 \le q$. To begin with, let $n \ge 3$ be odd and $m = n + 2$ (so m is odd). There is a Z_p action on $S^2 \times S^m$ with fixed set S^3. One can obtain such an action by letting η be the Hopf 2-plane bundle over S^2, $-\eta$ its inverse (i.e., $\eta \oplus -\eta$ = trivial 4-plane bundle). Let ε be a trivial $m-3$ plane bundle. $-\eta \oplus \varepsilon$ admits a fibre-wise orthogonal action of Z_p which leaves only the zero section (i.e., S^2) fixed. Consider $\eta \oplus (-\eta \oplus \varepsilon)$. Let Z_p act trivially on η. Taking unit sphere bundles yields an action of Z_p on $S^2 \times S^m$ ($m = n + 2$) with fixed set the total space of the sphere bundle of η (i.e., S^3) (see [3] for other such examples). Now remove a fixed point to obtain a Z_p action on a space Y which is homotopy equivalent to $S^2 \vee S^{n+2}$ and has contractible fixed set. Let Z_p act trivially on S^{n-3}, and take the join of S^{n-3} and Y. This space W has Z_p action with contractible fixed set and is itself homotopy equivalent to $S^n \vee S^{2n}$. Now let Z_p act on S^{3n} with fixed set S^q for q odd (e.g., take a linear action) and form the one-point union (at a fixed point) of W and S^{3n}. This provides all examples.

4. Proof of Theorem 2

By the Universal Coefficient Theorem, we have $H^{in}(X) = Z_p$ for $i = 0, 1, 2, 3$. We choose generators $v_i \in H^{in}(X)$, $i = 1, 2, 3$, respectively, satisfying the relations $v_1^2 = 0$ and $v_1 v_2 = 0$. Consider the Leray-Serre spectral sequence of the map $\pi: X_G \to BG$ with coefficients in the constant sheaf $\mathscr{H}^*(X)$ associated to $G = Z_p$ ($G = Z_p = \pi_1(BG)$ acts trivially on $H^*(X)$). The E_2-term of the spectral sequence is

$$E_2^{k,l} = H^k(BG) \otimes H^l(X).$$

Since X has no fixed points and p is odd, n must be odd and $E_2 \ne E_\infty$. So some differential:

$$d_r: E_r^{k,l} \to E_r^{k+r, l-r+1}$$

must be nontrivial. This is only possible for $r = n+1$, $2n+1$ and $3n+1$, and it is easily seen that

$$d_{n+1}(1 \otimes v_1) = d_{n+1}(1 \otimes v_3) = 0 \quad \text{and} \quad d_{n+1}(1 \otimes v_2) \ne 0.$$

So

$$E_r^{k,2n} = 0 = E_r^{k+n+1,n}$$

for all k and $r > n+1$. Obviously then, $d_{2n+1} = 0$.

If $d_{3n+1}(1 \otimes v_3) = 0$, then the bottom and top lines of the spectral sequence survive to infinity and this contradicts the fact that $H^*(X_G) = 0$ for $* > 3n$ (for G acts freely, hence $F_G = \varnothing$). Therefore $d_{3n+1}(1 \otimes v_3) \neq 0$ so that

$$E_\infty^{k,3n} = 0 = E_\infty^{k+3n+1,0}.$$

Hence we obtain

$$H^j(X_G) = \begin{cases} Z_p & \text{for } j > 3n, \\ Z_p & \text{for } 0 \leq j \leq n-1 \text{ and } 2n+1 \leq j \leq 3n, \\ Z_p \oplus Z_p & \text{for } n \leq j \leq 2n. \end{cases}$$

To determine the multiplicative structure, note that for $k \leq 3n$, $E_2^{k,0} \subset H^k(X_G)$. Let $x = s \otimes 1 \in E_\infty^{1,0}$ and $y = t \otimes 1 \in E_\infty^{2,0}$. Then $\pi^*(s) = x$, $\pi^*(t) = y$ and $\beta_p(x) = y$, by naturality of the Bockstein cohomology operation. The homomorphism

$$y \cup (\cdot) \colon E_\infty^{k,l} \to E_\infty^{k+2,l}$$

is an isomorphism for $k \leq 3n-2$ if $l = 0$ and for $k \leq n-2$ if $l = n$. Therefore multiplication by $y \in H^2(X_G)$

$$y \cup (\cdot) \colon H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism for $k \leq 2(n-1)$. The element $1 \otimes v_1 \in E_2^{0,n}$ is a permanent cocycle and determines an element w in $E_\infty^{0,n} \cdot E_2^{*,*}$, and $E_\infty^{*,*}$ are bigraded commutative algebras and therefore the total complex $\operatorname{Tot} E_\infty^{*,*}$ given by

$$(\operatorname{Tot} E_\infty^{*,*})^m = \bigoplus_{k+l=m} E_\infty^{k,l}$$

is a graded commutative algebra isomorphic to

$$Z_p[x, y, w]/(x^2, w^2, wy^{(n+1)/2}, y^{(3n+1)/2}).$$

Now we choose an element $z \in H^n(X_G)$ such that $i^*(z) = v_1$. Because the composition πi factors through a point, $i^*\pi^*$ is zero in positive degrees and hence we can assume that $zy^{(n+1)//2} = 0$. Since the multiplication by y is an isomorphism in degrees less than $2(n-1)$, $zy^i \neq 0$ for $2i < n-1$. Thus we have

$$H^*(X_G) = Z_p[x, y, z]/(x^2, z^2, zy^{(n+1)/2}, y^{3n+1)/2})$$

as graded commutative algebras. $\pi \colon X_G \to X/G$ is a homotopy equivalence and so induces a cohomology isomorphism. This completes the proof. □

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