COMPACT ELEMENTS AND SMALLEST FAITHFUL REPRESENTATION OF C*-ALGEBRAS

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Abstract. Let \( A \) be a \( C^* \)-algebra which either allows a faithful separable representation or is postliminal. We prove that \( A \) then admits a smallest faithful representation if and only if the ideal of compact elements is an essential ideal in \( A \).

Introduction

A faithful representation \((H, \varphi)\) of a \( C^* \)-algebra \( A \) will be called a smallest faithful representation of \( A \), if \((H, \varphi)\) is contained in every faithful representation of \( A \). It follows from [7], Theorem 5.1.5, that any two smallest faithful representations of a given \( C^* \)-algebra \( A \) are unitarily equivalent. The main result of this paper says that for \( C^* \)-algebras which allow a faithful separable representation and for postliminal \( C^* \)-algebras, there is a smallest faithful representation if and only if the ideal \( I \) of compact elements is an essential ideal, i.e. iff \( I \) has the annihilator \( \{0\} \) in the algebra. The "if" part of this result stays true for arbitrary \( C^* \)-algebras.

The proof of the "if" part can be outlined as follows: The ideal \( I \) of compact elements admits a smallest faithful representation \((H, \varphi_I)\). If \( I \) is an essential ideal, then the unique extension of \((H, \varphi_I)\) to \( A \) over \( H \) is a smallest faithful representation of \( A \).

The "only if" part of the proof is sketched below: Let \((K, \psi)\) be an irreducible representation of \( A \). Consider the ideal consisting of the elements of the annihilator of \( K \) which lie in the kernel of every irreducible representation not unitarily equivalent to \((K, \psi)\). This ideal is non-zero iff \((K, \psi)\) is contained in the smallest faithful representation \((H, \varphi)\) of \( A \). To every irreducible representation \((K, \psi)\) contained in \((H, \varphi)\) we can thus associate a simple ideal. Let \( F \) be the family of ideals defined in this way. The elements of \( F \) are mutually orthogonal. Since \( A \) admits a smallest faithful representation, the elements of \( F \) do so as well, and the intersection of the annihilators of the elements of \( F \) is the zero ideal. Thus, the ideal generated by the union of the elements of \( F \) is an essential ideal of \( A \). The elements of \( F \) consist entirely of compact elements of \( A \). In the separable case this is an immediate consequence of Rosenberg's
Theorem. It follows that the ideal generated by the union of the elements of $F$ is contained in $I$ and that $I$ is essential too.

I. Compact elements

1. Definitions and notational conventions. Let $I$ be a subset of the $C^*$-algebra $A$, and suppose $(H, \varphi)$ is a representation of $A$. Let $K$ be a linear subspace of $H$. The linear span of the set $\{\varphi(i)k \mid i \in I, k \in K\}$ is denoted by $\varphi(I)K$. We denote the closure of $\varphi(I)K$ by $\varphi(I)K^\perp$. $\varphi(I)H^\perp$ is called the essential subspace of $I$ under the representation $(H, \varphi)$. Let $B$ be a $C^*$-subalgebra of the $C^*$-algebra $A$, and suppose $K$ is a closed linear subspace of $H$, invariant for $B$. Then the restriction of $(H, \varphi)$ to $K$ and $B$ is denoted by $(H, \varphi)_{B, K}$. We denote $(H, \varphi)_{B, \varphi(B)H^\perp}$ simply by $(H, \varphi)_B$.

We write $(H, \varphi) \cong (K, \psi)$ to express the fact that $(H, \varphi)$ and $(K, \psi)$ are unitarily equivalent representations. If $(K, \psi)$ is a subrepresentation of $(H, \varphi)$ we write $(K, \psi) \subseteq (H, \varphi)$. $(H_1, \varphi_1) \subseteq (H_2, \varphi_2)$ and $(H_2, \varphi_2) \subseteq (H_1, \varphi_1)$ imply $(H_1, \varphi_1) \cong (H_2, \varphi_2)$ (see e.g. [7], Theorem 5.1.5).

An element $a$ of a $C^*$-algebra $A$ is said to be compact in $A$, if there is a faithful representation $(H, \varphi)$ of $A$ such that $\varphi(a)$ is a compact operator. We define the dimension of $a$ in $A$ to be the minimum rank the image of $a$ takes on under faithful representations of $A$. We denote this number by $\dim^a A$ and say that $a$ is finite dimensional in $A$ if $\dim^a A < \infty$. If $p$ and $q$ are Murray von Neumann equivalent projections of $A$ we write $p \sim q$. If we speak about an ideal of $A$, it is always meant to be a closed two-sided ideal.

We give a brief account of what needs to be known about compact elements of $C^*$-algebras in the sequel (see also [1], [5], [11], [12], [13]).

2. Proposition. Let $A$ be a $C^*$-algebra.

(a) Let $I$ be an ideal of $A$. If $(H, \varphi_I)$ is a non-degenerate representation of $I$, then there is a unique extension $(H, \varphi_A)$ of $(H, \varphi_I)$ defined on all of $A$.

If $(H, \varphi_I)$ is faithful, then $\ker(\varphi_A) = \text{Ann}(I)$. If, moreover, $I$ is essential, then $(H, \varphi_A)$ is faithful too.

(b) Suppose that $B$ is a hereditary subalgebra of $A$ and that $b \in B$. Then $\dim_B b = \dim_A b$ and $b$ is compact as an element of $B$ iff $b$ is compact as an element of $A$. In particular this applies in the special case where $B$ is an ideal.

(c) The set of compact elements of $A$ is an ideal. It is generated by the set of one-dimensional projections. The function $\dim_A$, defined on $A$, is invariant under Murray von Neumann Equivalence. The ideal generated by a one-dimensional projection of $A$ is simple. Equivalent one-dimensional projections generate the same ideal. The ideal of compact elements of $A$ is the restricted sum of the simple ideals obtained in this way.

(d) Let $I$ be the ideal of compact elements of $A$. To each simple subideal $I_k$ of $I$, one and only one equivalence class of irreducible representations of $I$, whose restriction to $I_k$ is non-trivial. This correspondence is bijective. But every representation of $I$ is a direct product of irreducible representations. Together with (c) this implies that $I$ admits a smallest faithful representation.

(e) Every faithful representation $(H, \varphi)$ of a $C^*$-algebra contains a quasi-equivalent, faithful subrepresentation, which maps every $n$-dimensional element to an $n$-dimensional operator and every compact element to a compact operator, simultaneously.
Proof. We must give a few hints on how to prove (c), (d) and (e).

A projection $p \in A$ is one-dimensional iff $pAp$ is a one-dimensional $C^*$-algebra.

To show the “if” part, let $\tau_p : A \to \mathbb{C}$ be defined by the equation $pap = \tau_p(a)p \ (a \in A)$. We call $\tau_p$ the associated state of $p$ on $A$. Now, $pAp$ is a hereditary subalgebra of $A$. A state on a hereditary subalgebra of $A$ allows exactly one extension to a state on $A$, and if the former is pure, the latter will be equally so (see e.g. [7], p. 91, Theorem 3.3.9; p. 148, Theorem 5.1.13). Thus $\tau_p$ is a pure state on $A$, and the representation $(H_{\tau_p}, \varphi_{\tau_p})$ of $A$ associated with the pure state $\tau_p$ is irreducible. We call $(H_{\tau_p}, \varphi_{\tau_p})$ the representation of $A$ associated with $p$. By use of the transitivity theorem of R. Kadison, it can be shown that $\text{rank } \varphi_{\tau_p}(p) = 1$. Moreover, if $q$ is a one-dimensional projection of $A$, then

$$\varphi_{\tau_p}(q) \neq 0 \iff p \sim q.$$ 

Now suppose $(H, \varphi)$ is a representation of $A$ such that $\varphi(p) \neq 0$. Then $(H_{\tau_p}, \varphi_{\tau_p}) \leq (H, \varphi)$. To see this, note that $x \in \text{im} (\varphi(p))$ and $\|x\| = 1$ imply $(\varphi(a)x|x) = \tau_p(a) \ (a \in A)$. Therefore, if $(H, \varphi)$ is faithful, then it contains every representation associated with one-dimensional projections of $A$. Take a quasiequivalent subrepresentation $(K, \psi)$ of $(H, \varphi)$, such that the multiplicity of every subrepresentation associated with a one-dimensional projection is one. $(K, \psi)$ is faithful, and $\text{rank } \psi(p) = 1$ for every projection $p$ with $\text{dim}_C pAp = 1$.

$(K, \psi)$ maps every element of the ideal of compact elements $I$ to a compact operator, since $I$ is generated by the one-dimensional projections of $A$ (see below). Moreover, any representation of $I$ is a direct product of representations associated with one-dimensional projections. Thus $I$ admits a smallest faithful representation.

$I$ is generated by the one-dimensional projections: Reduce the problem to the Hermitian case. Let thus $a \in I$ Hermitian and suppose that $(H, \varphi)$ is a faithful representation of $A$, such that $\varphi(a)$ is a compact operator. Then there is a strictly decreasing sequence $(\lambda_n)_N$ of positive real numbers with lower bound zero and a sequence $(p_n)_N$ of finite-dimensional projections in $B(H)$ such that the series $\sum_N \lambda_n \cdot p_n$ converges to $\varphi(a)$. Functional calculus shows that $p_n \in \varphi(A)$ for every $n \in N$. The problem is thus reduced to the case where $a$ is a finite-dimensional projection of $A$. Then $\text{dim}_C aAa < \infty$, i.e. $aAa$ is of the form $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ with $n_1, \ldots, n_k$ natural numbers.

It follows that $a = 1_{aAa}$ is a sum of $n_1 + \cdots + n_k$ one-dimensional projections.

The ideal generated by a one-dimensional projection $p$ is the simple ideal associated with the irreducible representation $(H_{\tau_p}, \varphi_{\tau_p})$ (see Introduction and Lemma II.3). \qed

II. Smallest faithful representations

II.1 Lemma. Let $A$ be a $C^*$-algebra allowing of a smallest faithful representation $(H, \varphi)$. Then $(H, \varphi)$ is unitarily equivalent to a certain direct sum of irreducible representations of $A$, every summand appearing with multiplicity one.

Proof. If $PS(A)$ denotes the set of pure states on $A$, then $\bigoplus_{PS(A)} (H, \varphi)$ is a direct sum of irreducible representations which is faithful and therefore...
contains \((H, \varphi)\). However, subrepresentations of a direct sum of irreducible representations are unitarily equivalent to direct sums of irreducible representations themselves. In particular, \((H, \varphi)\) is unitarily equivalent to a direct sum of irreducible representations. The statement about multiplicity one is evident. □

II.2 Proposition. Let \((I_\lambda)_\Lambda\) be a family of ideals of a \(C^*-\)algebra \(A\), such that the two following conditions hold:

(i) \(\bigcap_\Lambda \text{Ann}(I_\lambda) = \{0\}\).

(ii) For any two distinct elements \(\mu, \nu \in \Lambda\), \(I_\mu \cap I_\nu = \{0\}\).

Then \(A\) admits a smallest faithful representation iff \(I_\lambda\) admits a smallest faithful representation for every \(\lambda \in \Lambda\).

Proof. We start with the assumption that for each \(\lambda \in \Lambda\), there is a smallest faithful representation \((H_\lambda, \varphi_\lambda)\) of \(I_\lambda\). Let \((H_\lambda, \varphi_\lambda)\) denote its unique extension to \(A\). Then \(\text{ker}(\varphi_\lambda)\) equals \(\text{Ann}(I_\lambda)\). Therefore \(\bigcap_\Lambda \text{ker}(\varphi_\lambda) = \bigcap_\Lambda \text{Ann}(I_\lambda) = \{0\}\). But this means that the direct sum \(\bigoplus_\Lambda (H_\lambda, \varphi_\lambda)\) is faithful. Now, for a \(\lambda \in \Lambda\) and a faithful representation \((H, \varphi)\) of \(A\), the restriction of \((H, \varphi)\) to \(I_\lambda\) is faithful, and therefore contains the smallest faithful representation of \(I_\lambda\). It follows that \((H, \varphi)\) contains \((H_\lambda, \varphi_\lambda)\). From the fact that for distinct \(\mu, \nu \in \Lambda\) the essential spaces of \(I_\mu, I_\nu\) in \(H\) are mutually orthogonal, we conclude that \((H, \varphi)\) contains \(\bigoplus_\Lambda (H_\lambda, \varphi_\lambda)\). Hence, \(\bigoplus_\Lambda (H_\lambda, \varphi_\lambda)\) is a smallest faithful representation of \(A\).

We now assume that \(A\) admits a smallest faithful representation \((H, \varphi)\). Let \(\mu \in \Lambda\), and suppose \((H_\mu, \varphi_\mu)\) is a faithful non-degenerate representation of \(I_\mu\). Let \((H_\mu, \varphi_\mu)\) be the unique extension of \((H_\mu, \varphi_\mu)\) to \(A\). Then the faithful representation \((H_\mu, \varphi_\mu) \oplus (\bigoplus_{\lambda \in \Lambda, \lambda \neq \mu} (H, \varphi)_{(A, \varphi(I_\lambda)H^-)})\) of \(A\) contains \((H, \varphi) \cong \bigoplus_\Lambda (H, \varphi)_{(A, \varphi(I_\lambda)H^-)}\). But \((H, \varphi)_{(A, \varphi(I_\lambda)H^-)}\) and \((H, \varphi)_{(A, \varphi(I_\mu)H^-)}\) are disjoint (\(\lambda \neq \mu\)). This implies that \((H, \varphi)_{(A, \varphi(I_\lambda)H^-)} \leq (H_\mu, \varphi_\mu)\) and \((H, \varphi)_{(I_\mu, \varphi(I_\mu)H^-)} \leq (H_\mu, \varphi_\mu)\). \((H, \varphi)_{(I_\mu, \varphi(I_\mu)H^-)}\) is thus a smallest faithful representation of \(I_\mu\). □

II.3 Lemma. Let \((H, \varphi)\) be an irreducible representation of a \(C^*-\)algebra \(A\), and suppose \(I\) is the intersection of the kernels of the irreducible representations of \(A\), disjoint from \((H, \varphi)\). Then \(I\) is a simple ideal of \(A\). Moreover, if \(I\) is non-zero, then \(\text{Ann}(I)\) and \(\text{ker}(\varphi)\) coincide.

Proof. The proof of this lemma is easily found by the reader. □

II.4 Definition. We say that a \(C^*-\)algebra \(A\) is completely decomposable into its simple ideals, iff \(\bigcap_\Lambda \text{Ann}(I_\lambda) = \{0\}\) for a family \((I_\lambda)_\Lambda\) of simple ideals of \(A\). Note that \((I_\lambda)_\Lambda\) then contains every non-zero simple ideal and satisfies conditions (i) and (ii) of Proposition II.2.

II.5 Theorem. A \(C^*-\)algebra \(A\) has a smallest faithful representation iff the following conditions are satisfied:

(i) \(A\) is completely decomposable into its simple ideals.

(ii) Every simple ideal of \(A\) admits a smallest faithful representation.

Proof. Suppose \(A\) satisfies conditions (i) and (ii). Then Proposition II.2 applies to the family of simple ideals of \(A\), and it follows that \(A\) has a smallest faithful representation.

Conversely, suppose \(A\) admits a smallest faithful representation \((H, \varphi)\). Let \((H_\gamma, \varphi_\gamma)_\Gamma\) be a family containing exactly one representative of each equivalence
class of irreducible representations of $A$. $\exists \Omega \subset \Gamma$ s.t. $(H, \varphi) \cong \bigoplus_{\Omega} (H_\omega, \varphi_\omega)$ (see Lemma II.1). Let $I_\mu := \bigcap_{\Gamma \setminus \{\mu\}} \ker(\varphi_\gamma)$ ($\mu \in \Gamma$). Then $I_\mu$ is a simple ideal ($\mu \in \Gamma$) (see Lemma II.3). Consider $\Lambda := \{\lambda \in \Gamma | I_\lambda \neq \{0\}\}$. Then $\mu \in \Omega$ $\Leftrightarrow \bigoplus_{\Gamma \setminus \{\mu\}} (H_\gamma, \varphi_\gamma)$ is not faithful $\Leftrightarrow \bigcap_{\Gamma \setminus \{\mu\}} \ker(\varphi_\gamma) \neq \{0\} \Leftrightarrow \mu \in \Lambda$, i.e. $\Lambda = \Omega$.

Now, $\bigcap_{\Lambda} \Ann(I_\lambda) = \bigcap_{\Omega} \ker(\varphi_\lambda) = \{0\}$, since $\ker \varphi_\lambda = \Ann(I_\lambda)$ ($\lambda \in \Lambda$) (see Lemma II.3). This implies (i).

(ii) follows from Proposition II.2 and from (i). $\blacksquare$

The next lemma is a corollary of Rosenberg's Theorem (see e.g. [6], p. 505).

II.6 Lemma. Let $A$ be a simple $C^*$-algebra, faithfully representable over a separable Hilbert space and allowing of a smallest faithful representation. Then there is a separable Hilbert space $H$ such that $A$ is $*$-isomorphic to $K(H)$, the algebra of compact operators over $H$.

Proof. By simplicity of $A$, every non-zero representation of $A$ is faithful and contains the smallest faithful representation of $A$. Therefore, every non-zero irreducible representation of $A$ is unitarily equivalent to the smallest faithful representation of $A$, hence is unique to within unitary equivalence. Since $A$ is assumed to be separable too, Rosenberg's Theorem applies to $A$. Consequently, $A$ is $*$-isomorphic to $K(H)$ for some separable Hilbert space $H$. $\blacksquare$

II.7 Theorem. Let $A$ be a $C^*$-algebra for which there is a faithful separable representation. Then $A$ admits a smallest faithful representation if and only if the ideal of compact elements of $A$ is an essential ideal of $A$.

Proof. Suppose the ideal $I$ of compact elements of $A$ is essential. The family $\{I\}$ of ideals satisfies conditions (i) and (ii) of Proposition II.2, and by Proposition I.2(d) $I$ has a smallest faithful representation. It follows from Proposition II.2 that $A$ allows a smallest faithful representation.

On the other hand, if $A$ admits a smallest faithful representation, then, by Theorem II.5, $A$ is decomposable into its simple ideals $(I_\lambda)_\Lambda$, all of which admit a smallest faithful representation. Since $A$ can be represented faithfully over a separable Hilbert space, the smallest faithful representations of the $I_\lambda$ act on separable Hilbert spaces as well. It follows from Lemma II.6 that $I_\lambda$ consists entirely of elements which are compact in $I_\lambda$ ($\lambda \in \Lambda$). But the ideals $I_\lambda$ are hereditary subalgebras of $A$, hence their elements are also compact in $A$, i.e. $I_\lambda \subset I$ (see Proposition 1.2(b)). Then $\Ann(I_\lambda) \supset \Ann(I)$ ($\lambda \in \Lambda$). It follows that $\{0\} = (\bigcap_{\Lambda} \Ann(I_\lambda)) \supset \Ann(I)$. $I$ is therefore an essential ideal of $A$. $\blacksquare$

II.8 Theorem. A postliminal $C^*$-algebra $A$ admits a smallest faithful representation if and only if the ideal of compact elements is an essential ideal of $A$.

Proof. The "if" part of the proof of Theorem II.7 remains valid for general $C^*$-algebras. A fortiori it applies to postliminal $C^*$-algebras.

Conversely, if $A$ admits a smallest faithful representation, then, in complete analogy to the "only if" part of the proof of Theorem II.7, it suffices to show that the simple ideals $I_\lambda$ of $A$ consist entirely of elements which are compact in $I_\lambda$. By simplicity of $I_\lambda$, the smallest faithful representation $(H_\lambda, \psi_\lambda)$ of $I_\lambda$ is irreducible ($\lambda \in \Lambda$). Let $(H_\lambda, \varphi_\lambda)$ be the extension of $(H_\lambda, \psi_\lambda)$ to $A$. 

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Then, since $(H, \varphi)$ is non-zero on $I_\lambda$ and since $A$ is postliminal, we have: 
${\{0\} \neq \varphi(I_\lambda) \cdot K(H_\lambda) \subset K(H_\lambda) \subset \varphi(A)$.

Therefore, $\varphi^{-1}(\varphi(I_\lambda) \cdot K(H_\lambda))$ is a non-zero ideal in $I_\lambda$. Then $\varphi^{-1}(\varphi(I_\lambda) \cdot K(H_\lambda)) = I_\lambda$, since $I_\lambda$ is simple, and $\psi(I_\lambda) \subset K(H_\lambda)$. $\varphi$ being faithful implies that every element of $I_\lambda$ is compact in $I_\lambda$. □

II.9 Historical remark and conclusion. In his paper [8] Naimark rose the question of whether a $C^*$-algebra which admits a unique irreducible representation is *-isomorphic to the algebra of compact operators over some Hilbert space. Using a partial result provided by Naimark in [8], A. Rosenberg gave a proof of the above statement in the separable case in [10], p. 529. However, as mentioned by Pedersen in [9], p. 255, and by Fell and Doran in [6], p. 506, the general case remains an open problem till today. If Rosenberg's Theorem can be generalized to a class of $C^*$-algebras closed under the taking of subideals, then Theorem II.7 immediately generalizes to this class of $C^*$-algebras.

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