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ON SOME IDEALS OF NEST ALGEBRAS

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ABSTRACT. The purpose of this paper is to characterize the relations of some ideals of a nest algebra such as the Jacobson radical $R_{\mathcal{N}}$, Larson strong radical $R_{\mathcal{N}}^{\infty}$, and ideal $\mathcal{G}$, and to describe all the ideals between $R_{\mathcal{N}}$ and $\mathcal{G}$.

1. PRELIMINARIES

Let $\mathcal{N}$ be a nest (i.e., a totally ordered complete set of projections containing O and I which is closed in the strong operator topology) on a separable Hilbert space H. We denote by Alg$\mathcal{N}$ the algebra of all operators in $B(H)$ that leave invariant every element of $\mathcal{N}$. A nest $\mathcal{N}$ is continuous of its core, which is the von Neumann algebra generated by the elements of $\mathcal{N}$, is a nonatomic von Neumann algebra. $K(H)$ denotes the set of all the compact operators in $B(H)$. We shall call a set $\{P_\alpha | \alpha \in \Lambda\}$ of intervals $P_\alpha = M_\alpha - N_\alpha$, $M_\alpha, N_\alpha \in \mathcal{N}$, $N_\alpha < M_\alpha$, of the nest $\mathcal{N}$, a partition of $\mathcal{N}$ if the intervals are pairwise orthogonal and the sum $\sum_{\alpha\in\Lambda} P_\alpha = I$ in the strong topology. Since H is a separable Hilbert space, then every partition is denumerable. Given a nest $\mathcal{N}$, Larson strong radical $R_{\mathcal{N}}^{\infty}$ is the collection of all operators X in Alg$\mathcal{N}$ for which, given $\varepsilon > 0$, there exists a partition $\{P_\alpha | \alpha \in \Lambda\}$ of $\mathcal{N}$ such that $\|P_\alpha X P_\alpha\| < \varepsilon$ for all $\alpha \in \Lambda$. If we restrict all partitions to be finite sets of intervals, we will obtain the Jacobson radical $R_{\mathcal{N}}$ of Alg$\mathcal{N}$ [4]. It is clear that $R_{\mathcal{N}}^{\infty}$ always contain $R_{\mathcal{N}}$. The ideals $R_{\mathcal{N}}$, $R_{\mathcal{N}}^{\infty}$ have been studied by some authors ([3], [4]). In this paper, we will study the ideal $\mathcal{G}$ of Alg$\mathcal{N}$ and its relations with $R_{\mathcal{N}}$ and $R_{\mathcal{N}}^{\infty}$. Terms and notation not defined here are taken from [4].

Definition 1. If E_1, E_2 are nonzero orthogonal intervals from $\mathcal{N}$ such that $E_1 B(H) E_2 \subseteq$ Alg$\mathcal{N}$, we say E_1, E_2 are strictly ordered and write $E_1 \ll E_2$.

A mutually orthogonal family of intervals is said to be strictly ordered if it is linearly ordered by the relation $\ll$. The length of such a family is its cardinality.

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Definition 2. If $A \in \operatorname{Alg}\mathcal{N}$ and $\varepsilon > 0$, we define the ε-order of A to be the number

$R_\varepsilon(A) = \operatorname{Sup}\{n \mid$ there exists a strictly ordered family $\mathcal{F}$ of length n with $\|EAE\| \geq \varepsilon$ for all $E \in \mathcal{F}\}$.

Definition 3. Let

$$\mathcal{G} = \{A \in \operatorname{Alg}\mathcal{N} \mid R_\varepsilon(A) < +\infty, \text{ for every } \varepsilon > 0\},$$
$$\mathcal{G}_u = \{A \in \operatorname{Alg}\mathcal{N} \mid \sup_{\varepsilon>0} R_\varepsilon(A) < +\infty\},$$
$$R^\alpha_{\mathcal{N}} = \{A \in \operatorname{Alg}\mathcal{N} \mid EAE = 0 \text{ for every atom } E \text{ of } \mathcal{N}\}.$$

It is easy to verify that $\mathcal{G}$, $\mathcal{G}_u$, $R^\alpha_{\mathcal{N}}$ are ideals of $\operatorname{Alg}\mathcal{N}$. □

2. Main results

In the paper [1], it is shown that $\mathcal{G}$ is a norm-closed ideal of $\operatorname{Alg}\mathcal{N}$ which contains $R_{\mathcal{N}}$. In this section, we will study the relations of $\mathcal{G}$, $\mathcal{G}_u$, $R_{\mathcal{N}}$, $R^\infty_{\mathcal{N}}$ and $R^\alpha_{\mathcal{N}}$.

Let $x \otimes y$ denote the rank-one operator acting on H such that $x \otimes y(z) = \langle z, x\rangle y$, for all $z \in H$.

Theorem 1. *$\mathcal{G}_u$ is closed in the norm topology if and only if $\mathcal{N}$ is a finite nest.*

Proof. If $\mathcal{N}$ is finite, then $\mathcal{G}_u = \mathcal{G} = \operatorname{Alg}\mathcal{N}$. Thus $\mathcal{G}_u$ is norm-closed.

Now suppose $\mathcal{G}_u$ is a norm-closed ideal; we claim that $\mathcal{N}$ is finite. Otherwise, there exists $P \in \mathcal{N}$ such that $P = P_-$ $(P \neq 0)$ or $P = P_+$ $(P \neq I)$. We only need to consider the case: $P = P_-$ $(P \neq 0)$. Choose $\{P_n\}$, $P_n \in N$, such that $P_n < P_{n+1}$, $P = \lim_{n\to\infty} P_n$ in the strong operator topology. Let $E_n = P_{n+1} - P_n$, and $x_n \in E_n$ such that $\|x_n\| = 1$, $n = 1, 2, \ldots$.

Let $T_n = \sum_{k=1}^n \frac{1}{k^2} x_{k+1} \otimes x_k$, and $T = \sum_{k=1}^\infty \frac{1}{k^2} x_{k+1} \otimes x_k$; then $\lim_{n\to\infty} T_n = T$ in the norm topology and $T_n \in \mathcal{G}_u$ $(n = 1, 2, \ldots)$. Now we prove that $T \notin \mathcal{G}_u$. For arbitrary positive integer M, there exists $\varepsilon > 0$ such that $\varepsilon < \frac{1}{(2M-1)^2}$. Let $F_k = P_{2k+1} - P_{2k-1}$ $(k = 1, 2, \ldots, M)$, thereby, $\|F_k T F_k\| = \frac{1}{(2k-1)^2} \geq \frac{1}{(2M-1)^2} > \varepsilon$ $(k = 1, 2, \ldots, M)$, and $R_\varepsilon(T) \geq M$. Since M is arbitrary, we have $\sup_{\varepsilon>0} R_\varepsilon(T) = +\infty$. Thus $T \notin \mathcal{G}_u$, which contradicts the closedness of $\mathcal{G}_u$.

Theorem 2. *Let $\mathcal{N}$ be a nest. Then*

(1) $R^\alpha_{\mathcal{N}} \cap \mathcal{G} = R^\infty_{\mathcal{N}} \cap \mathcal{G} = R_{\mathcal{N}}$.

(2) $\mathcal{G} = R_{\mathcal{N}}$ *if and only if $\mathcal{N}$ is continuous.*

(3) $\mathcal{G} \subsetneq R^\infty_{\mathcal{N}}$ *if and only if $\mathcal{N}$ is continuous.*

(4) $\mathcal{G} \supsetneq R^\infty_{\mathcal{N}}$ *if and only if $\mathcal{N}$ is finite.*

For arbitrary nest $\mathcal{N}$, $\mathcal{G} \neq R^\infty_{\mathcal{N}}$.

Proof. (1) The chain $R_{\mathcal{N}} \subseteq R^\infty_{\mathcal{N}} \cap \mathcal{G} \subseteq R^\alpha_{\mathcal{N}} \cap \mathcal{G}$ is evident. It remains to show that each $X \in R^\alpha_{\mathcal{N}} \cap \mathcal{G}$ belongs to $R_{\mathcal{N}}$. Apply Ringrose's Criterion, using the seminorms $i^+_N(X)$ and $i^-_N(X)$ (see [4]). Let $\varepsilon > 0$ and $N \in \mathcal{N}$. By symmetry it suffices to show that $i^+_N(X) \leq \varepsilon$. If $N^+ > N$, then $N^+ - N$ is an atom and

$$i^+_N(X) = \|(N^+ - N)X(N^+ - N)\| = 0.$$

Otherwise there is a sequence N_n decreasing to N in $\mathcal{N}$. Suppose that $i^+_N(X) > \varepsilon$. For each fixed k, $(N_k - N_n)X(N_k - N_n)$ converges strongly to

$(N_k - N)X(N_k - N)$, which is greater than ε in norm. So, by the strong lower semicontinuity of the norm there is a k' such that

$$\|(N_k - N_{k'})X(N_k - N_{k'})\| > \varepsilon.$$

Passing to a subsequence, we can assume

$$\|(N_k - N_{k+1})X(N_k - N_{k+1})\| > \varepsilon$$

for all k. But this means that $R_\varepsilon(X) = +\infty$, which is a contradiction.

(2) It is easy to prove that $E \in \mathscr{G}$, $E \notin R^\alpha_{\mathscr{N}}$, for every atom E. By (1), $\mathscr{G} = R_{\mathscr{N}}$ if and only if $\mathscr{G} \subseteq R^\alpha_{\mathscr{N}}$ if and only if $\mathscr{N}$ is continuous.

For (3), (4), by (1), we can prove them easily.

Proposition 1. *Let $\mathscr{N}$ be a nest. Then*

(1) $R_{\mathscr{N}} \subseteq \overline{\mathscr{G}_u}$, *where the closure is in the norm topology.*

(2) $R_{\mathscr{N}} \subset \mathscr{G}_u$ *if and only if $\mathscr{N}$ is finite.*

(3) $\mathscr{G}_u \subset R_{\mathscr{N}}$ *if and only if $\mathscr{N}$ is continuous.*

For arbitrary nest $\mathscr{N}$, $R_{\mathscr{N}} \neq \mathscr{G}_u$.

Proof. (1) Since $R_{\mathscr{N}} = \overline{\text{span}}\{PAP^\perp | P \in \mathscr{N}, A \in \text{Alg}\mathscr{N}\}$, where the closure is in the norm topology, and $PAP^\perp \in \mathscr{G}_u$, for every $P \in \mathscr{N}$, $A \in \text{Alg}\mathscr{N}$, (1) holds.

(2) If $\mathscr{N}$ is finite, then $R_{\mathscr{N}} \subset \mathscr{G}_u = \text{Alg}\mathscr{N}$. If $\mathscr{N}$ is infinite, we have constructed $T \in R_{\mathscr{N}}$, but $T \notin \mathscr{G}_u$ in the proof of Theorem 1. This contradicts $R_{\mathscr{N}} \subset \mathscr{G}_u$.

(3) If $\mathscr{N}$ is continuous, from Theorem 2 (2), $\mathscr{G} = R_{\mathscr{N}} \supset \mathscr{G}_u$. If $\mathscr{N}$ is not continuous, then there exists an atom E. Let $x, y \in E$ such that $\|x\| = \|y\| = 1$; then $x \otimes y \notin R_{\mathscr{N}}$, $x \otimes y \in \mathscr{G}_u$. Thus $\mathscr{G}_u \not\subseteq R_{\mathscr{N}}$.

In the paper [2], Lance gave a standard form for all ideals containing the radical of a nest algebra, but not in a very explicit manner. In the last theorem of this paper, we will, by the index function of atoms, characterize all the norm-closed ideals between $R_{\mathscr{N}}$ and $\mathscr{G}$ in an explicit way.

Let $\{E_\alpha | \alpha \in \Lambda\}$ be all the atoms of $\mathscr{N}$. Let

$$\text{ind}(\alpha) = \begin{cases} 0 & \text{if } \dim E_\alpha < \infty, \\ 1 & \text{if } \dim E_\alpha = \infty \end{cases}$$

denote the index function of $\{E_\alpha | \alpha \in \Lambda\}$.

Let $\mathscr{F} = \{\varphi | \varphi \colon \Lambda \to \{0, 1, 2\}$ such that $\varphi(\alpha) \le \text{ind}(\alpha) + 1\}$. For $\varphi \in \mathscr{F}$, $\alpha \in \Lambda$, let

$$M_{\varphi(\alpha)} = \begin{cases} 0 & \text{if } \varphi(\alpha) = 0, \\ E_\alpha K(H) E_\alpha & \text{if } \varphi(\alpha) = 1, \\ E_\alpha B(H) E_\alpha & \text{if } \varphi(\alpha) = 2. \end{cases}$$

Definition 4. For $\varphi \in \mathscr{F}$, let

$$\mathscr{D}_\varphi = \left\{ T \,\middle|\, \begin{array}{l} T = \sum_{\alpha \in \Lambda} T_\alpha, T_\alpha \in M_{\varphi(\alpha)}, \\ \lim \|T_\alpha\| = 0 \end{array} \right\},$$

where $\lim \|T_\alpha\| = 0$ means: for every $\varepsilon > 0$, set $\{\alpha \in \Lambda | \, \|T_\alpha\| \ge \varepsilon\}$ is finite, and let

$$\mathscr{G}_\varphi = R_{\mathscr{N}} \oplus \mathscr{D}_\varphi.$$

In the next theorem, we will identify $\mathscr{G}$ with $R_{\mathscr{N}} \oplus D_0$, where D_0 is the set of operators which are norm convergent sums of the form $\sum E_i T E_i$ (with E_i an enumeration of the atoms). Clearly, $D_0 = D_{\varphi_0}$, where

$$\varphi_0(\alpha) = \begin{cases} 1 & \text{if } \operatorname{ind}(\alpha) = 0, \\ 2 & \text{if } \operatorname{ind}(\alpha) = 1. \end{cases}$$

Theorem 3. *Let $\mathscr{N}$ be a nest. Then*

(1) *If I is a closed two-sided ideal of* $\operatorname{Alg}_{\mathscr{N}}$ *in the norm topology and $R_{\mathscr{N}} \subseteq I \subseteq \mathscr{G}$, then there exists $\varphi \in \mathscr{F}$ such that $I = \mathscr{G}_\varphi$. In particular, $\mathscr{G} = R_{\mathscr{N}} \oplus \mathscr{D}_0$.*

(2) *For every $\varphi \in \mathscr{F}$, $\mathscr{G}_\varphi$ is a norm-closed two-sided ideal of* $\operatorname{Alg}\mathscr{N}$, *and $R_{\mathscr{N}} \subseteq \mathscr{G}_\varphi \subseteq \mathscr{G}$.*

(3) *$\mathscr{G}_\varphi = \mathscr{G}_{\varphi'}$ if and only if $\varphi = \varphi'$.*

Proof. (1) For every $\alpha \in \Lambda$, $E_\alpha I E_\alpha$ is an ideal of $E_\alpha(\operatorname{Alg}\mathscr{N})E_\alpha$, which is closed in the norm topology. If $\operatorname{ind}(\alpha) = 0$, then $E_\alpha I E_\alpha = 0$ or $E_\alpha I E_\alpha = E_\alpha B(H) E_\alpha = E_\alpha K(H) E_\alpha$. If $\operatorname{ind}(\alpha) = 1$, then $E_\alpha I E_\alpha = 0$ or $E_\alpha I E_\alpha = E_\alpha K(H) E_\alpha$ or $E_\alpha I E_\alpha = E_\alpha B(H) E_\alpha \neq E_\alpha K(H) E_\alpha$. Let

$$\varphi(\alpha) = \begin{cases} 0 & \text{if } E_\alpha I E_\alpha = 0, \\ 1 & \text{if } E_\alpha I E_\alpha = E_\alpha K(H) E_\alpha, \\ 2 & \text{if } E_\alpha I E_\alpha = E_\alpha B(H) E_\alpha \neq E_\alpha K(H) E_\alpha. \end{cases}$$

Then $\varphi \in \mathscr{F}$ and $M_{\varphi(\alpha)} \subseteq I$. By the definition of $\mathscr{G}_\varphi$, we have $\mathscr{G}_\varphi \subseteq I$. On the other hand, for $T \in I$, let $S = T - \sum_{\alpha\in\Lambda} T_\alpha$, where $T_\alpha = E_\alpha T E_\alpha$, $\alpha \in \Lambda$. Since $T \in \mathscr{G}$, we have $\sum_{\alpha\in\Lambda} T_\alpha \in \mathscr{G}$ and $\lim \|T_\alpha\| = 0$. Therefore $S \in \mathscr{G}$. Furthermore $E_\alpha S E_\alpha = 0$, for $\alpha \in \Lambda$; then $S \in R^\alpha_{\mathscr{N}}$ and so $S \in R^\alpha_{\mathscr{N}} \cap \mathscr{G} = R_{\mathscr{N}}$. Thus $T = S + \sum_{\alpha\in\Lambda} T_\alpha \in \mathscr{G}_\varphi$. Since T is arbitrary, we have $I \subseteq \mathscr{G}_\varphi$, which implies that $I = \mathscr{G}_\varphi$.

(2) Letting $\varphi \in \mathscr{F}$, by the definition of $\mathscr{G}_\varphi$, we know that $R_{\mathscr{N}} \subseteq \mathscr{G}_\varphi \subseteq \mathscr{G}$. Let I be the norm-closed two-sided ideal of $\operatorname{Alg}\mathscr{N}$ generated by $\mathscr{G}_\varphi$. Then $R_{\mathscr{N}} \subseteq I \subseteq \mathscr{G}$. By the proof of (1), we have $I = \mathscr{G}_\varphi$. Thus $\mathscr{G}_\varphi$ is a two-sided ideal of $\operatorname{Alg}\mathscr{N}$, which is closed in the norm topology.

(3) If $\varphi \neq \varphi'$, then there exists $\alpha \in \Lambda$ such that $\varphi(\alpha) \neq \varphi'(\alpha)$. Thus $M_{\varphi(\alpha)} \neq M_{\varphi'(\alpha)}$, and therefore, by a trivial argument, $\mathscr{G}_\varphi \neq \mathscr{G}_{\varphi'}$.

The following corollary follows Theorem 3 immediately.

Corollary 1. *Let s denote the cardinality of set $\{I | R_{\mathscr{N}} \subseteq I \subseteq \mathscr{G}, I$ is a norm closed two-sided ideal of* $\operatorname{Alg}\mathscr{N}\}$. *Then*

(1) *If* $\operatorname{ind}(\alpha) = 0$, *for every $\alpha \in \Lambda$, then $s = 2^\Lambda$, where 2^Λ denotes the cardinality of set which consists of all subsets of Λ.*

(2) *If* $\operatorname{ind}(\alpha) = 1$, *for every $\alpha \in \Lambda$, then $s = 3^\Lambda$, where 3^Λ denotes the cardinality of the set which consists of all maps from Λ to $\{0, 1, 2\}$. Particularly, if $\overline{\Lambda} = n$ (i.e. the cardinality of Λ) and* $\operatorname{ind}(\alpha) = 1$ *for every $\alpha \in \Lambda$, then $s = 3^n$.*

(3) *If $\overline{\Lambda} = n$, $\Lambda_0 = \{\alpha | \operatorname{ind}(\alpha) = 0\}$, $\Lambda_1 = \{\alpha | \operatorname{ind}(\alpha) = 1\}$, and $\overline{\Lambda_0} = m$, $\overline{\Lambda_1} = n - m$, then $s = 2^m \cdot 3^{n-m}$.*

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