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THE TATE CONJECTURE FOR t-MOTIVES

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ABSTRACT. A version of the Tate conjecture is proved for φ-modules of " t-motive type".

In this note, we formulate a version of the Tate conjecture for φ-modules, and give a proof of it in a special but essential case. Similar results have been obtained independently by Tamagawa [3] in a more general setting.

Let K be an algebraic function field in one variable over a finite field, whose field of constants is $\mathbb{F}_q$, π a place of K, and K_π the completion of K at π. Let k be any field containing $\mathbb{F}_q$. We set $K_k := k \otimes_{\mathbb{F}_q} K$, and denote by $K_{k,\pi}$ the completion of K_k with respect to the π-adic topology (these may not be fields). Let σ be the endomorphism (q-th power Frobenius of k) $\otimes$ (id_K) of K_k, and also its natural extension to $K_{k,\pi}$. By a *φ-module* (M, φ) (or simply M) over K_k (resp. over $K_{k,\pi}$), we mean a free K_k-module (resp. $K_{k,\pi}$-module) M of finite rank equipped with a σ-semi-linear map $\varphi : M \to M$. Morphisms of φ-modules are defined naturally. Tensor products $M \otimes N$ (with diagonal φ-action) exist. Internal homs $\mathrm{Hom}(M, N)$ (with φ : "$f \mapsto \varphi_N \circ f \circ \varphi_M^{-1}$") may or may not exist. For a φ-module M, let M^φ denote the fixed part of M by φ. This is a K-subspace (resp. K_π-subspace) of M. If the internal hom $H = \mathrm{Hom}(M, N)$ exists, then H^φ is the space $\mathrm{Hom}_\varphi(M, N)$ of φ-module homomorphisms of M to N. Now the Tate conjecture in our context is

Conjecture. Let M be a φ-module over K_k. If k is of finite type over $\mathbb{F}_q$, then the natural map of K_π-vector spaces

$$K_\pi \otimes_K M^\varphi \ \to \ (K_{k,\pi} \otimes_{K_k} M)^\varphi$$

is an isomorphism.

In general, this is not true (cf. [3]).

Equivalently, by fixing a K_k-basis of M, the conjecture can be stated also as follows: Let A be a matrix in $\mathrm{M}_r(K_k)$. Consider the linear Frobenius equation

$$(*) \qquad AX^\sigma = X,$$

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where the indeterminate X is considered in $K_{k,\pi}^{\oplus r}$, on which σ acts componentwise. Let V (resp. V_π) be the space of solutions of (*) in $K_k^{\oplus r}$ (resp. $K_{k,\pi}^{\oplus r}$). Then the natural map of K_π-vector spaces $K_\pi \otimes_K V \to V_\pi$ is an isomorphism.

These problems can (and should) be considered also with K_k replaced by certain localizations of it.

The injectivity is easy to see, so the essence is in the surjectivity.

A reduction can be made: suppose K' is a subfield of K which contains $\mathbb{F}_q$ and over which K is finite. Let π' be the restriction of π to K'. A φ-module M over K_k can be regarded as a φ-module over K'_k. If the conjecture is true for (K', π', M), then it is also true for (K, π, M) (use the identification $K'_{\pi'} \otimes_{K'} K = \prod_{\pi|\pi'} K_\pi$, etc.). So we may and do assume $K = \mathbb{F}_q(t)$ and identify π with a monic irreducible element of $\mathbb{F}_q[t]$. Replacing K again by the subfield $\mathbb{F}_q(\pi)$, we may assume $\pi = t$ (so K_k is the polynomial ring $k[t]$ to which the inverses of all monic polynomials in $\mathbb{F}_q[t]$ have been adjoined, and $K_{k,\pi} = k((t))$).

Now we prove the conjecture assuming that k is a function field in one variable over $\mathbb{F}_q$ (this is not essential) and that M comes from t-motives of characteristic different from π, by which we *mean* the following (cf. [1]): there exist a non-zero element θ of k and positive integers d and d' such that the map φ is represented with respect to some K_k-basis of M by a matrix A of the form

$$A = (t-\theta)^d B^{-1},$$

where B is a matrix in $\mathrm{M}_r(k[t])$ with $\det B$ of the form $u(t-\theta)^{d'}$, $u \in k^\times$ (so we allow any $A \in \mathrm{GL}_r(k[t, \frac{1}{t-\theta}])$ in (*), which may not be in $\mathrm{M}_r(K_k)$).

We will show that, if the equation (*) has a solution $\underline{\hat{x}}$ in $K_{k,\pi}^{\oplus r}$, then it has a solution $\underline{x}$ in $K_k^{\oplus r}$ which is sufficiently close (in the t-adic= π-adic topology) to $\underline{\hat{x}}$ (so that, if $(\underline{\hat{x}}_i)$ is a basis for V_π, so is $(\underline{x}_i)$ for V). By assumption, we have

$$(**) \qquad (t-\theta)^d \underline{\hat{x}}^\sigma = B\underline{\hat{x}}, \qquad \theta \neq 0.$$

Write $(t-\theta)^d = \sum_{i=0}^d \theta_i t^i$ (with $\theta_i \in k$); $B = \sum_{i=0}^N B_i t^i$ (with $B_i \in \mathrm{M}_r(k)$); and $\underline{\hat{x}} = \sum_{i\geq 0} x_i t^i$ (with $x_i \in k^{\oplus r}$). Then (**) yields

$$(***) \qquad \theta_0 x_i^\sigma + \cdots + \theta_d x_{i-d}^\sigma = B_0 x_i + \cdots + B_N x_{i-N}, \qquad i \geq 0.$$

(Here negatively indexed terms are zero.) For any valuation v of k, let $v(x_i)$ denote the minimum of the valuations of the entries of x_i, and $v(B)$ the minimum of the valuations of the entries of B_i for all $i \geq 0$. If $v(\theta) \leq 0$, then

$$v(x_i) \geq \min\{v(x_{i-1}), \cdots, v(x_{i-d}), \frac{v(x_i)+v(B)}{q}, \cdots, \frac{v(x_{i-N})+v(B)}{q}\},$$

so we see recursively that $v(x_i) \geq v(B)/(q-1)$ for all $i \geq 0$. If $v(\theta) > 0$, we replace X in (**) by $\theta^{-e}X$ (resp. B by $\theta^{e(q-1)}B$) for some e to have $v(\theta^{e(q-1)}B) \geq 0$. Then Anderson's arguments[1] in §4 of [2] imply the

[1] His arguments there show in particular the following: let $\mathcal{O}$ be the integer ring of the completion k_v of k at the place v. Let B be a matrix in $\mathrm{M}_r(\mathcal{O}[[t]])$ such that $\det B = u(t-\theta)^{d'}$ with a non-zero $u \in \mathcal{O}$. Then any solution X to the equation $(t-\theta)^d X^\sigma = BX$ in $k_v[[t]]^{\oplus r}$ is

holomorphy of the new solution $\theta^e \underline{\hat{x}}$, hence the old solution $\underline{\hat{x}}$ satisfies $v(x_i) \geq -ev(\theta)$ for all $i \geq 0$. Thus the values $v(x_i)$, $i \geq 0$, are bounded below for all valuations v of k, by constants which are non-negative for almost all v. By the bounded height theorem, there appear in fact only finitely many x_i's in the sequence $x_0, x_1, \cdots$. Accordingly, there appear only finitely many equations (***), and one can choose a "periodic" (except for finitely many terms) solution $\underline{x} = \sum x_i' t^i$ in $K_{k,\pi}^{\oplus r}$ to (***) closely enough to the original $\underline{\hat{x}}$. Periodicity implies that $\underline{x}$ is rational with denominator in $\mathbb{F}_q[t]$, hence $\underline{x} \in K_k^{\oplus r}$.

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automatically in $\mathscr{O}[[t]]^{\oplus r}$. (The proof is found in the last page of [2], except that we need to use his Lemma 7 for a more general $\Phi = B$ as above, with the maximal ideal $\mathscr{M}^{\mathrm{sep}}$ in the statement replaced by the integer ring $\mathscr{O}^{\mathrm{sep}}$ of a separable closure of k_v. This can be proved easily by looking at each term of the t-adic expansion of the given equation, just as we did in (***).)