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A NOTE ON G-INVARIANT FORMS

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ABSTRACT. If G is a finite group, the reduction $\bmod p$ of a module supporting a nondegenerate G-invariant form need not itself support such a form. However, under a suitable hypothesis on the splitting field (quadratic closure) and a carefully chosen lattice within the module (for reduction $\bmod p$), this will always be the case. The argument given is elementary and self-contained.

Let $K \supseteq R \to F$ be a "p-modular system" for the finite group G. That is, R is a valuation domain with maximal ideal P, field of fractions K, and residue class field F where K and F are splitting fields for G in characteristics 0 and p respectively. Thus, every (finitely generated) KG-module V contains an RG-lattice V_0, that is, an R-free, finitely generated RG-submodule of V which generates V as a K-space. Necessarily, $\operatorname{rank}_R V_0 = \dim_K V$ holds for any such V_0. Suppose now V is an irreducible KG-module affording the "real-valued" character χ (that is, $\chi(g) = \chi(g^{-1})$ for $g \in G$). Then V supports a nondegenerate symmetric or alternating G-invariant form $(\ ,\ )$ and it is natural to ask whether some reduction $\bmod p$ has this property. Specifically, this asks for the existence of an RG-lattice $V_0 \subseteq V$ with R-basis $\{v_i\}$ so that the Gram matrix $((v_i, v_j))$ has entries in R and has determinant a unit of R. This last condition of course implies that $(\ ,\ )$ induces a nondegenerate G-invariant form on the FG-module $\overline{V}_0 = V_0/PV_0$.

Let $\mathrm{OChar}(G)$ and $\mathrm{OBr}(G)$ denote the set of ordinary characters and Brauer characters afforded by G-modules supporting a nondegenerate symmetric bilinear form. Similarly, define $\mathrm{SkChar}(G)$ and $\mathrm{SkBr}(G)$ for the case of alternating forms. Let χ° denote the restriction of $\chi \in \mathrm{Char}(G)$ to p-regular elements. A consequence of an affirmative answer to the question of the first paragraph is that the function $\chi \mapsto \chi^\circ$ sends $\mathrm{OChar}(G)$ to $\mathrm{OBr}(G)$ and $\mathrm{SkChar}(G)$ to $\mathrm{SkBr}(G)$. However, these weaker assertions are already proven in [3] when R is a discrete valuation domain. (The argument is also outlined in [4].)

In fact, the proof in the DVR case amounts to finding an RG-lattice V_0 in V so that $(\ ,\ )$ restricted to V_0 takes on values in R and induces a form on $\overline{V}_0$ whose radical $\overline{V}_0^{\perp}$ relative to this induced form is an FG-submodule which possesses a *nondegenerate* G-invariant form of the same type as $(\ ,\ )$.

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The question of the first paragraph essentially asks whether V_0 can always be chosen as above so that $\overline{V}_0^{\perp} = 0$. The answer depends on the choice of p-modular system, which the main result of this paper shows.

Theorem. *Let $K \supseteq R \to F$ be a p-modular system for G with K quadratically closed. If V is any KG-module and $(\ , \ )$ is a nondegenerate symmetric or alternating G-invariant form on V, then an RG-lattice $V_0 \subseteq V$ can always be found satisfying $(V_0, V_0) \subseteq R$ and $\overline{V}_0^{\perp} = 0$.*

The result above is not always true (nor even desirable) for other choices of p-modular systems. For example, in establishing the basic structure of a p-block of defect 1 (in particular, in establishing the uniserial nature of RG-lattices contained in simple KG-modules) it appears necessary to work with representations over "p-minimal" fields (as defined in Goldschmidt's book [2]). In that setting, suppose $\overline{V}_0$ is the reduction mod p of an RG-lattice in a simple module in a block of defect 1, and assume $\overline{V}_0$ supports a nondegenerate G-invariant form. Suppose also $\overline{V}_0$ has two real-valued irreducible Brauer constituents. If $M \leq \overline{V}_0$ is a maximal locally isotropic submodule, then $M^{\perp}/M$ is semisimple of composition length 2, violating the uniseriality of $\overline{V}_0$. (Explicit examples are easy to construct. Let $G = S_3$ be the smallest nonabelian group and $p = 3$, and let V be the QG-module corresponding to the faithful irreducible character. In this case, $R \subseteq G$ is the ring of 3-local integers. Then $\overline{V}_0$ can never support a nondegenerate G-invariant form for any choice of $V_0 \subseteq V$.)

On the other hand, there are settings in which it is desirable to find $V_0 \subseteq V$ satisfying $(V_0, V_0) \subseteq R$ and $\overline{V}_0^{\perp} = 0$. For example, if p is odd and if $\mathrm{O}(W)$ denotes the full orthogonal group of a vector space over K or F relative to some nondegenerate symmetric form, then $\mathrm{O}(W)$ has a double cover. Hence, if $\rho\colon G \to \mathrm{O}(W)$ is any orthogonal representation of G, then the pull-back along ρ determines a double cover of G. The cohomology class of this cover depends only on the character afforded by W, and we have functions $h\colon \mathrm{OChar}(G) \to H^2(G, \{\pm 1\})$ and $h\colon \mathrm{OBr}(G) \to H^2(G, \{\pm 1\})$. If now $\chi \in \mathrm{OChar}(G)$ is afforded by V and if $V_0 \subseteq V$ can be chosen so that $(V_0, V_0) \subseteq R$ with $\overline{V}_0^{\perp} = 0$, then the equality $h(\chi) = h(\chi^{\circ})$ becomes transparent. This result is Theorem 6.2 of [1], except now (with the right hypothesis on the p-modular system) the existence of the special lattice $V_0 \subseteq V$ is already guaranteed and need not appear as a hypothesis.

Following [4], it is convenient to introduce the following notation. Let

$$\mathscr{L}(V) = \{V_0 | V_0 \text{ is an } R\text{-sublattice of } V\}$$

where V is a finite-dimensional K-space. If $(\ , \ )$ is a nondegenerate form on V the collection of integral lattices is defined as

$$\mathscr{L}_{\mathrm{int}}(V) = \{V_0 \in \mathscr{L}(V) | (V_0, V_0) \subseteq R\},$$

while if V is also a KG-module, set

$$\mathscr{L}_G(V) = \{V_0 \in \mathscr{L}(V) | V_0 \text{ is } G\text{-invariant}\}.$$

Finally, set

$$\mathscr{L}_{G,\mathrm{int}}(V) = \mathscr{L}_{\mathrm{int}}(V) \cap \mathscr{L}_G(V).$$

In the presence of a nondegenerate form, the dual L^* of a lattice L in $\mathscr{L}(V)$ is defined by $L^* = \{x \in V | (x, L) \subseteq R\}$ and, by considering dual bases, $L^* \in \mathscr{L}(V)$. Evidently $L \subseteq L^*$ if and only if $L \in \mathscr{L}_{\text{int}}(V)$.

For the purposes of this note, call a lattice $L \in \mathscr{L}_{\text{int}}(V)$ "c-homogeneous" if the Gram matrix of this form computed with respect to some (hence any) R-basis of L has the form $c \cdot U$ for some nonsingular matrix U over R whose inverse also has entries in R. Notice that by consideration of dual bases we have $L^* = \frac{1}{c}L$ for c-homogeneous lattices (and L^* is $\frac{1}{c}$-homogeneous).

It is convenient to start with an elementary lemma (which is essentially the Gram-Schmidt process).

Lemma. *Let* $(\ , \ )$ *be a nondegenerate form defined on the* K*-space* V, *and let* L *be any integral* R*-sublattice of* V. *Then there exists an orthogonal decomposition of* L *as*

$$L = W_1 \perp W_2 \perp \cdots \perp W_k$$

where each W_i *is* c_i*-homogeneous for some* $c_i \neq 0$ *and* $R \supseteq (c_1) \supseteq (c_2) \supseteq \cdots \supseteq (c_k)$.

Proof. The set $(L, L) \subseteq R$ is an ideal of R which is finitely generated (by the entries of the Gram matrix with respect to some basis) and so it is principally generated, say by $c_1 \in R$. Let $(\ , \ )'$ denote the form $\frac{1}{c_1}(\ , \ )$. Then $(\ , \ )'$ induces a nonzero form on $\overline{L} = L/PL$, whose radical $\overline{L}^{\perp}$ therefore is proper in $\overline{L}$. Write $d = \dim_F \overline{L}/\overline{L}^{\perp}$ and choose $v_1, v_2, \ldots, v_d \in L$ so that the natural image of $\{v_1, v_2, \ldots, v_d\}$ in $\overline{L}/L^{\perp}$ is an F-basis. If $W_1 = \sum_{i=1}^{d} Rv_i$, then the original form restricted to W_1 is nondegenerate, and W_1 is c_1-homogeneous by construction. Thus, if $\{v_1^*, v_2^*, \ldots, v_d^*\}$ denotes the dual basis of $\{v_1, v_2, \ldots, v_d\}$ in KW_1, then $v_i^* \in \frac{1}{c_1}W_1$ for every i. Hence, if $x \in L$, then $(x, v_i)v_i^* \in W_1$ for each i, so that $x' = x - \sum (x, v_i)v_i^*$ belongs to L and $(x', W_1) = 0$. This shows $L = W_1 \perp Y$ where $Y = \{x \in L | (x, W_1) = 0\}$.

This construction may be continued in the R-submodule Y, producing the decomposition of the lemma, as required. □

Notice that if $L = W_1 \perp W_2 \perp \cdots \perp W_k$ as in the lemma, then

$$L^* = \frac{1}{c_1}W_1 \perp \frac{1}{c_2}W_2 \perp \cdots \perp \frac{1}{c_k}W_k.$$

It is possible to refine the decomposition so that $\operatorname{rank}_R(W_i) \leq 2$ for all i and even $\operatorname{rank}_R(W_i) = 1$ in the symmetric case when p is odd.

Proof of Theorem. If $L \in \mathscr{L}_{G,\text{int}}(V)$ define $r(L)$ to be the rank over F of the matrix obtained from the Gram matrix of $(\ , \ )$ by reducing the entries mod P. Select $L \in \mathscr{L}_{G,\text{int}}(V)$ so that $r(L)$ is maximal, and let $L = W_1 \perp W_2 \perp \cdots \perp W_k$ be the decomposition of L as in the previous lemma. We may clearly assume, using the notation of that lemma, that each inclusion $(c_{i+1}) \subset (c_i)$ is strict. If $(c_1) \neq R$, then $r(L) = 0$, $\frac{1}{\sqrt{c_1}}L \in \mathscr{L}_{G,\text{int}}(V)$ and $r(\frac{1}{\sqrt{c_1}}L) > 0$, for a contradiction. If $(c_k) \neq R$ let $V_1 = L + \sqrt{c_k}L^* \in \mathscr{L}_G(V)$. Then $V_1 = X_1 \perp X_2 \perp \cdots \perp X_k$, where

$$X_i = W_i + \frac{\sqrt{c_k}}{c_i}W_i = \begin{cases} W_i & \text{if } \sqrt{c_k} \in (c_i), \\ \frac{\sqrt{c_k}}{c_i}W_i & \text{if } \sqrt{c_k} \notin (c_i). \end{cases}$$

Clearly,

$$(X_i, X_i) = (W_i, W_i) = (c_i) \subseteq R \quad \text{if } \sqrt{c_k} \in (c_i)$$

while

$$(X_i, X_i) = \left(\frac{\sqrt{c_k}}{c_i} W_i, \frac{\sqrt{c_k}}{c_i} W_i \right) = \frac{c_k}{c_i^2}(c_i) \subseteq R \quad \text{if } \sqrt{c_k} \notin (c_i).$$

This proves that $V_1 \in \mathscr{L}_{G,\mathrm{int}}(V)$, and at the same time $(X_i, X_i) = R$ for both $i = 1$ and k. As each X_i is homogeneous we conclude that $r(V_1) \geq \mathrm{rank}_R(X_1) + \mathrm{rank}_R(X_k) > \mathrm{rank}_R(X_1) = r(L)$, for another contradiction.

Thus $(c_i) = R$ for all i, proving the theorem. □

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