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ALGEBRAIC AND TRIANGULAR n-HYPONORMAL OPERATORS

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ABSTRACT. In this paper we shall prove that if an operator $T \in \mathscr{L}(\bigoplus_1^n \mathbf{H})$ is a finite triangular operator matrix with hyponormal operators on main diagonal, then T is subscalar. As corollaries we get the following:

(1) Every algebraic operator is subscalar.
(2) Every operator on a finite-dimensional complex space is subscalar.
(3) Every triangular n-hyponormal operator is subscalar.

1. INTRODUCTION

Let $\mathbf{H}$ and $\mathbf{K}$ be separable, complex Hilbert spaces and $\mathscr{L}(\mathbf{H}, \mathbf{K})$ denote the space of all linear, bounded operators from $\mathbf{H}$ to $\mathbf{K}$. If $\mathbf{H} = \mathbf{K}$, we write $\mathscr{L}(\mathbf{H})$ in place of $\mathscr{L}(\mathbf{H}, \mathbf{K})$. An operator T in $\mathscr{L}(\mathbf{H})$ is called hyponormal if $TT^* \leq T^*T$ or, equivalently, if $\|T^*h\| \leq \|Th\|$ for each h in $\mathbf{H}$.

A linear bounded operator S on $\mathbf{H}$ is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi\colon C_0^m(\mathbf{C}) \to \mathscr{L}(\mathbf{H})$$

such that $\Phi(z) = S$, where as usual z stands for the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ stands for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace.

This paper is divided into four sections. Section 2 deals with some preliminary facts. In section 3, we shall state the Putinar theorem. In section 4, we shall prove our main theorem and several corollaries.

2. PRELIMINARIES

An operator $T \in \mathscr{L}(\mathbf{H})$ is said to satisfy the single-valued extension property if for any open subset U in $\mathbf{C}$, the function

$$z - T\colon O(U, \mathbf{H}) \to O(U, \mathbf{H})$$

defined by the obvious pointwise multiplication is one-to-one where $O(U, \mathbf{H})$ denotes the Fréchet space of $\mathbf{H}$-valued analytic functions on U with respect to

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uniform topology. If, in addition, the above function $z - T$ has closed range on $O(U, \mathbf{H})$, then T satisfies the Biship's condition (β).

In other terms, condition (β) means that, for any open set U and any sequence of analytic functions $f_n \in O(U, \mathbf{H})$, $\lim_{n\to\infty} f_n = 0$ in $O(U, \mathbf{H})$ whenever $\lim_{n\to\infty}(z - T)f_n = 0$. In particular, $(z - T)g = 0$ if and only if $g = 0$, where $g \in O(U, \mathbf{H})$.

2.1 **Lemma** ([MP], Theorem 5.5). *Every hyponormal operator has property* (β).

Let z be the coordinate in the complex plane $\mathbf{C}$, and let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space $\mathbf{H}$ and a bounded (connected) open subset U of $\mathbf{C}$. We shall denote by $L^2(U, \mathbf{H})$ the Hilbert space of measurable functions $f: U \to \mathbf{H}$, such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 d\mu(z)\right)^{1/2} < \infty.$$

The space of functions $f \in L^2(U, \mathbf{H})$ which are analytic functions in U (i.e., $\overline{\partial} f = 0$) is denoted by

$$A^2(U, \mathbf{H}) = L^2(U, \mathbf{H}) \cap O(U, \mathbf{H}).$$

$A^2(U, \mathbf{H})$ is called the Bergman space for U. Note that $A^2(U, \mathbf{H})$ is complete (i.e., $A^2(U, \mathbf{H})$ is a Hilbert space).

Let P denote the orthogonal projection of $L^2(U, \mathbf{H})$ onto $A^2(U, \mathbf{H})$. Let $L^\infty(U, \mathbf{H})$ denote the Banach space of essentially bounded $\mathbf{H}$-valued functions on U. Let $\overline{U}$ be the closure in $\mathbf{C}$ of the open set U, and let $C^p(\overline{U}, \mathbf{H})$ denote the space of continuously differentiable functions on $\overline{U}$ of order p, $0 \le p \le \infty$.

Cauchy-Pompeiu formula. For a bounded disk D,

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(s)}{s - z} ds - \frac{1}{\pi}\int_D \frac{\overline{\partial} f(s)}{s - z} d\mu(s)$$

where $z \in D$ and $f \in C^2(\overline{D}, \mathbf{H})$.

Remark. The function $g(z) = \int_{\partial D} \frac{f(s)}{s-z} d\mu(s)$ appearing in the Cauchy-Pompeiu formula is analytic in D and continuous on $\overline{D}$, in particular $g \in A^2(D, \mathbf{H})$ for $f \in C^2(\overline{D}, \mathbf{H})$.

Let us now define a special Sobolev type space. Let U again be a bounded open subset of $\mathbf{C}$ and m be a fixed non-negative integer. The vector-valued Sobolev space $W^m(U, \mathbf{H})$ with respect to $\overline{\partial}$ and of order m will be the space of those functions $f \in L^2(U, \mathbf{H})$ whose derivatives $\overline{\partial} f, \ldots, \overline{\partial}^n f$ in the sense of distributions still belong to $L^2(U, \mathbf{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\overline{\partial}^i f\|^2_{2,U}$$

$W^m(U, \mathbf{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathbf{H})$.

Let U be a (connected) bounded open subset of $\mathbf{C}$, and let m be a non-negative integer. The linear operator M of multiplication by z on $W^m(U, \mathbf{H})$ is continuous and has a spectral distribution of order m, defined by the relation

$$\Phi_M: C_0^m(\mathbf{C}) \to \mathscr{L}(W^m(U, \mathbf{H})), \qquad \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m. Let

$$V: W^m(U, \mathbf{H}) \to \bigoplus_0^m L^2(U, \mathbf{H})$$

be the operator $V(f) = (f, \overline{\partial} f, \ldots, \overline{\partial}^m f)$. Then V is an isometry such that $VM_z = (\bigoplus_0^m M_z)V$. Therefore, M_z is a subnormal operator.

3. Putinar's theorem

Let T be in $\mathscr{L}(\mathbf{H})$. Then for a given open bounded subset D of $\mathbf{C}$, $z - T$ acts (linearly and) continuously on the space $W^2(D, \mathbf{H})$.

3.1 **Lemma** ([Pu], Lemma 1.1). *If U and V are bounded connected open sets in $\mathbf{C}$, and if V is relatively compact in U, then there is a constant $c > 0$, such that*

$$\|f\|_{\infty, V} \leq c\|f\|_{2, U}$$

for every f in $A^2(U, \mathbf{H})$.

3.2 **Proposition** ([Pu], Proposition 2.1). *For a bounded disk D in complex plane there is a constant C_D, such that for an arbitrary operator T in $\mathscr{L}(\mathbf{H})$ and f in $W^2(D, \mathbf{H})$ we have*

$$\|(I - P)f\|_{2, D} \leq C_D(\|(z - t)^* \overline{\partial} f\|_{2, D} + \|(z - T)^* \overline{\partial}^2 f\|_{2, D})$$

where P denotes the orthogonal projection of $L^2(D, \mathbf{H})$ onto the Bergman space $A^2(D, \mathbf{H})$.

3.3 **Corollary** ([Pu], Corollary 2.2). *If T is hyponormal, then*

$$\|(I - P)f\|_{2, D} \leq C_D(\|(z - T)\overline{\partial} f\|_{2, D} + \|(z - T)\overline{\partial}^2 f\|_{2, D}).$$

3.4 **Theorem** ([Pu], Theorem 1). *Any hyponormal operator is subscalar of order 2.*

Proof. Let T be a hyponormal operator on the Hilbert space $\mathbf{H}$. Consider an arbitrary bounded open subset D of $\mathbf{C}$ and the quotient space

$$H(D) = \frac{W^2(D, \mathbf{H})}{\mathrm{cl}(z - T)W^2(D, \mathbf{H})}$$

endowed with the Hilbert space norm. The class of a vector f or an operator A on this quotient will be denoted by $\tilde{f}$, respectively $\tilde{A}$.

Note that M, the operator of multiplication by z on $W^2(D, \mathbf{H})$, leaves invariant $\mathrm{ran}(z - T)$, hence $\widetilde{M}$ is well defined.

On the other hand, the map

$$\Phi: C_0^2(\mathbf{C}) \to \mathscr{L}(W^2(D, \mathbf{H})), \qquad \Phi(f) = M_f$$

is a spectral distribution for M, of order 2. Thus the operator M is C^2-scalar. Since $\overline{\mathrm{ran}(z - T)}$ is invariant under every operator M_f, $f \in C_0^2(\mathbf{C})$, we infer that $\widetilde{M}$ is a C^2-scalar operator with spectral distribution Φ.

Define

$$V: \mathbf{H} \to \frac{W^2(D, \mathbf{H})}{\mathrm{cl}(z - T)W^2(D, \mathbf{H})}$$

by $V(h) = 1\tilde{\otimes}h$ where $1 \otimes h$ denotes the constant h. Then

$$VT = \widetilde{M}V.$$

Indeed, $VTh = (1 \otimes Th)^{\sim} = (z\otimes h)^{\sim} = \widetilde{M}(1 \otimes h)^{\sim} = \widetilde{M}Vh$. In particular $\operatorname{ran}V$ is an invariant subspace for $\widetilde{M}$. In order to conclude the proof of this theorem, it is enough to show the following lemma.

3.5 **Lemma** ([Pu], Lemma 2.3). *Let D be a bounded disk which contains $\sigma(T)$. Then the operator V is one-to-one and has closed range.*

Proof. We have to prove the following assertion: If h_n in $\mathbf{H}$ and f_n in $W^2(D, \mathbf{H})$ are sequences such that

$$\lim_{n\to\infty} \|(z-T)f_n + h_n\|_{W^2} = 0, \tag{1}$$

then $\lim_{n\to\infty} h_n = 0$. The assumption (1) implies

$$\lim_{n\to\infty} (\|(z-T)\overline{\partial} f_n\|_{2,D} + \|(z-T)\overline{\partial}^2 f_n\|_{2,D}) = 0.$$

By Corollary 3.3,

$$\lim_{n\to\infty} \|(I-P)f_n\|_{2,D} = 0.$$

Then by (1),

$$\lim_{n\to\infty} \|(z-T)Pf_n + h_n\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n\to\infty} \|Pf_n(z) + (z-T)^{-1}h_n\| = 0$$

uniformly by the preceding consequence of Proposition 3.2. Hence,

$$\left\| \frac{1}{2\pi i}\int_\Gamma Pf_n(z)dz + h_n \right\| \to 0.$$

But $\int_\Gamma Pf_n dz = 0$. Hence, $\lim_{n\to\infty} h_n = 0$. □

4. Main theorems

In this section, we shall prove that every algebraic and triangular n-hyponormal operator is subscalar.

4.1 **Definition.** An operator $T \in \mathscr{L}(\mathbf{H})$ is algebraic if there is a non-zero polynomial p such that $p(T) = 0$.

4.2 **Definition.** An operator $T \in \mathscr{L}(\mathbf{H})$ is nilpotent if $T^n = 0$ for some integer n.

4.3 **Proposition.** *Every nilpotent operator is an algebraic operator.*

An interesting characterization of algebraic operators was given by P. R. Halmos.

4.4 **Theorem** ([Ha]). *If T is an algebraic operator and p is a polynomial of minimal degree n such that $p(T) = 0$, then T is unitarily equivalent to a finite operator matrix of the form*

$$\begin{pmatrix} \alpha_1 & T_{12} & \dots & \dots & T_{1n} \\ 0 & \alpha_2 & T_{23} & \dots & \dots \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \alpha_{n-1} & T_{n-1,n} \\ 0 & \dots & \dots & 0 & \alpha_n \end{pmatrix}$$

where α_i are the roots of the polynomial p.

The following theorem will be proved in this paper.

4.5 **Theorem.** *If an operator $T \in \mathscr{L}(\bigoplus_1^n \mathbf{H})$ is a finite operator matrix of the form*

$$T = \begin{pmatrix} T_{11} & T_{12} & \dots & \dots & T_{1n} \\ 0 & T_{22} & T_{23} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & T_{n-1,n-1} & T_{n-1,n} \\ 0 & \dots & \dots & 0 & T_{n,n} \end{pmatrix}$$

where $T_{i,i}$ are hyponormal for $i = 1, 2, \dots, n$, then T is a subscalar operator of order $2n$.

Proof. Consider an arbitrary bounded open subset D of $\mathbf{C}$ which contains $\sigma(T)$ and the quotient space

$$H(D) = \frac{\oplus_1^n W^{2n}(D, \mathbf{H})}{\operatorname{cl}(T - z) \oplus_1^n W^{2n}(D, \mathbf{H})}$$

endowed with the Hilbert space norm. Let M_z be the multiplication operator with z on $W^{2n}(D, \mathbf{H})$. Then $\bigoplus_1^n M_z$ is a C^{2n}-scalar subnormal operator and its spectral distribution is

$$\Phi\colon \bigoplus_1^n C_0^{2n}(\mathbf{C}) \to \mathscr{L}(\bigoplus_1^n W^{2n}(D, \mathbf{H})), \qquad \Phi(\bigoplus_1^n f_i) = \bigoplus_1^n M_{f_i}.$$

Since M_z commutes with M_{T-z}, $\bigoplus_1^n \widetilde{M}_z$ is still a scalar operator of order $2n$, with $\widetilde{\Phi}$ as spectral distribution.

Let $\bigoplus_1^n V$ be the operator

$$(\bigoplus V)(\bigoplus h_i) = (1 \otimes h_1, \dots, 1 \otimes h_n)^t + \overline{(T - z) \oplus_1^n W^{2n}(D, \mathbf{H})},$$

from $\bigoplus_1^n \mathbf{H}$ into $H(D)$, denoting by $(1 \otimes h_1, \dots, 1 \otimes h_n)^t$ the constant function $\bigoplus_1^n h_i$. Then

$$(\bigoplus_1^n V)T = (\bigoplus_1^n \widetilde{M}_z)(\bigoplus_1^n V)$$

In particular, $\operatorname{ran}(\bigoplus_1^n V)$ is an invariant subspace for $\bigoplus_1^n \widetilde{M}_z$. In order to conclude the proof of this theorem, it is enough to show Lemma 4.6.

4.6 **Lemma.** *Let D be a bounded disk which contains $\sigma(T)$. Then the operator $\bigoplus_1^n V\colon \bigoplus_1^n \mathbf{H} \to H(D)$ is one-to-one and has closed range.*

Proof of Lemma 4.6. Let $\bigoplus_1^n h_i^k \in \bigoplus_1^n \mathbf{H}$ and $\bigoplus_1^n f_i^k \in \bigoplus_1^n W^{2n}(D, \mathbf{H})$ be sequences (in k) such that

$$\lim_{k\to\infty} \|(T - z)\bigoplus_1^n f_i^k + \bigoplus_1^n (1 \otimes h_i^k)\|_{\oplus_1^n W^{2n}} = 0.$$

It suffices to show that $\lim_{k\to\infty} \bigoplus_1^n h_i^k = 0$. The limit given directly above can be written as

$$\begin{cases} \lim_{k\to\infty} \|(T_{11} - z)f_1^k + T_{12}f_2^k + \cdots + T_{1n}f_n^k + 1 \otimes h_1^k\|_{W^{2n}} = 0, \\ \quad \vdots \\ \lim_{k\to\infty} \|(T_{j,j} - z)f_j^k + T_{j,j+1}f_{j+1}^k + \cdots + T_{j,n}f_n^k + 1 \otimes h_j^k\|_{W^{2n}} = 0, \\ \quad \vdots \\ \lim_{k\to\infty} \|(T_{n,n} - z)f_n^k + 1 \otimes h_n^k\|_{W^{2n}} = 0. \end{cases}$$

In order to prove Lemma 4.6 we need the following:

Fact. For $t = 1, 2, 3, \ldots, n$,

$$\begin{cases} (1,1) & \lim_{k\to\infty} \|(T_{11} - z)f_1^k + T_{12}f_2^k + \cdots + T_{1t}f_t^k + 1 \otimes h_1^k\|_{W^{2t}} = 0, \\ & \quad \vdots \\ (1,j) & \lim_{k\to\infty} \|(T_{j,j} - z)f_j^k + T_{j,j+1}f_{j+1}^k + \cdots + T_{j,t}f_t^k + 1 \otimes h_j^k\|_{W^{2t}} = 0, \\ & \quad \vdots \\ (1,t) & \lim_{k\to\infty} \|(T_{t,t} - z)f_t^k + 1 \otimes h_t^k\|_{W^{2t}} = 0. \end{cases}$$

We prove this fact by induction. We assume that Lemma 4.7 holds for some given $t = 2, 3, \ldots, n$. We only need to verify that

$$\begin{cases} \lim_{k\to\infty} \|(T_{11} - z)f_1^k + T_{12}f_2^k + \cdots + T_{1,t-1}f_{t-1}^k + 1 \otimes h_1^k\|_{W^{2(t-1)}} = 0, \\ \quad \vdots \\ \quad \vdots \\ \lim_{k\to\infty} \|(T_{t-1,t-1} - z)f_{t-1}^k + 1 \otimes h_{t-1}^k\|_{W^{2(t-1)}} = 0. \end{cases}$$

However the reader will note that this result follows directly from $(1, 1), \ldots, (1, t)$ provided $\lim_{k\to\infty} \|\overline{\partial}^i f_t^k\|_{2,D} = 0$ for $i = 0, 1, \ldots, 2(t-1)$. So this will be shown to be true.

Claim 1. $\lim_{k\to\infty} h_t^k = 0$.

The proof of Lemma 3.5 suitably modified to include the higher order partials with $T = T_{t,t}$ shows the claim to be true.

Claim 2. $\lim_{k\to\infty} \|(I - P)\overline{\partial}^i f_t^k\|_{2,D} = 0$ for $i = 0, \ldots, 2(t-1)$.

By Claim 1 and the equation $(1, t)$, $\lim_{k\to\infty} \|(T_{t,t} - z)f_t^k\|_{W^{2t}} = 0$. Then we can apply Proposition 3.2 and Corollary 3.3 with $T = T_{t,t}$. In fact,

$$\begin{aligned}
&\|(I-P)(f_t^k, \overline{\partial} f_t^k, \dots, \overline{\partial}^{2t-2} f_t^k)\|_{2,D} \\
&\quad \le C_D(\|(T_{t,t}-z)^*(\overline{\partial} f_t^k, \dots, \overline{\partial}^{2t-1} f_t^k)\|_{2,D} \\
&\qquad + \|(T_{t,t}-z)^*(\overline{\partial}^2 f_t^k, \dots, \overline{\partial}^{2t} f_t^k\|_{2,D}) \\
&\quad \le C_D(\|(T_{t,t}-z)(\overline{\partial} f_t^k, \dots, \overline{\partial}^{2t-1} f_t^k)\|_{2,D} \\
&\qquad + \|(T_{t,t}-z)(\overline{\partial}^2 f_t^k, \dots, \overline{\partial}^{2t} f_t^k)\|_{2,D}),
\end{aligned}$$

where P denotes the orthogonal projection of $\bigoplus_{2t-1} L^2(D, \mathbf{H})$ onto the Bergman space $\bigoplus_{2t-1} A^2(D, \mathbf{H})$. Thus Claim 2 follows from $(1, t)$.

By Claim 2, for $i = 0, 1, \dots, 2(t-1)$

$$\lim_{k\to\infty} \|(T_{t,t}-z)\overline{\partial}^i f_t^k - (T_{t,t}-z)P\overline{\partial}^i f_t^k\|_{2,D} = 0.$$

Since $\lim_{k\to\infty} \|(T_{t,t}-z) f_t^k\|_{W^{2t}} = 0$ by Claim 1 and equation $(1, t)$,

$$\lim_{k\to\infty} \|(T_{t,t}-z)P\overline{\partial}^i f_t^k\|_{2,D} = 0$$

for $i = 0, 1, \dots, 2(t-1)$. Since every hyponormal operator has property (β) by Lemma 2.1, for $i = 0, 1, \dots, 2(t-1)$, $P\overline{\partial}^i f_t^k \to 0$ uniformly on compact subsets of D.

Consider $\sigma(T) \subset B(0, r) \subset \overline{B(0, r)} \subset D$. For $i = 0, 1, \dots, 2(t-1)$,

$$\begin{aligned}
\|P\overline{\partial}^i f_t^k\|_{2,D}^2 &= \int_D \|P\overline{\partial}^i f_t^k(z)\|^2 d\mu(z) \\
&= \int_{\overline{B(0,r)}} \|P\overline{\partial}^i f_t^k(z)\|^2 d\mu(z) + \int_{D\setminus\overline{B(0,r)}} \|P\overline{\partial}^i f_t^k(z)\|^2 d\mu(z).
\end{aligned}$$

By property (β), the first integral converges to 0. Since $T - z$ is invertible on $D\setminus\overline{B(0, r)}$, the second integral also converges to 0. Therefore,

$$\lim_{k\to\infty} \|P\overline{\partial}^i f_t^k\|_{2,D} = 0.$$

From Claim 2, we get $\lim_{k\to\infty} \|\overline{\partial}^i f_t^k\|_{2,D} = 0$ for $i = 0, 1, \dots, 2(t-1)$. So this completes the proof of the fact stated above.

Let us come back now to the proof of Lemma 4.6. By the fact, we get the equation

$$\lim_{k\to\infty} \|(T_{11}-z) f_1^k + 1 \otimes h_1^k\|_{W^2} = 0.$$

By the application of Lemma 3.5 with $T = T_{11}$, we get $\lim_{k\to\infty} h_1^k = 0$. Since $\lim_{k\to\infty} h_j^k = 0$ for $j = 1, \dots, n$, $\lim_{k\to\infty} h^k = 0$ where $h^k = (h_1^k, \dots, h_n^k)$. Thus $\bigoplus_1^n V$ is one-to-one and has closed range. □

This also concludes the proof of Theorem 4.5, because $\operatorname{ran}(\bigoplus_1^n V)$ is a closed invariant subspace for the scalar operator $(\bigoplus_1^n \widetilde{M}_z)$. □

4.8 **Corollary.** *If T is an algebraic operator, then T is a subscalar operator.*

Proof. It is clear from Theorem 4.4 and Theorem 4.5. □

4.9 **Corollary.** *Every operator on a finite-dimensional complex space is subscalar.*

4.10 **Definition.** An operator $T \in \mathscr{L}(\mathbf{H})$ is said to have property (α) if for every (not necessarily strict) contraction A, every operator X with dense range such that $XA = TX$, and every h in $\mathbf{H}$, there exists a non-zero polynomial p such that $p(T)h \in \operatorname{ran} X$.

4.11 **Lemma** ([ABFP], Proposition 3.9). *If a strict contraction T (i.e., $\|T\| < 1$) has the property (α), then T is an algebraic operator.*

4.12 **Corollary.** *If a strict contraction T has property (α), then T is subscalar.*

4.13 **Corollary.** *If $A \prec T$ and T is algebraic, then A is subscalar.*

Proof. By hypothesis, we can show A is algebraic. □

Remark. Let

$$B = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$$

where T is hyponormal. Then B is a non-hyponormal, but B is a subscalar operator of order 4.

4.14 **Definition.** An operator T in $\mathscr{L}(\bigoplus_1^n \mathbf{H})$ is said to be a triangular n-hyponormal operator if

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & \cdots & T_{1n} \\ 0 & T_{22} & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & T_{n-1,n-1} & T_{n-1,n} \\ \cdots & \cdots & \cdots & 0 & T_{nn} \end{pmatrix}$$

where (T_{ij}) are commuting hyponormal operators on $\mathbf{H}$.

4.15 **Corollary.** *Let T be a triangular n-hyponormal operator. Then T is a subscalar operator.*

4.16 **Question.** *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

where $\{T_i\}$ are commuting hyponormal operators. Is T subscalar?

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