SEQUENCES IN THE RANGE OF A VECTOR MEASURE
WITH BOUNDED VARIATION

CÁNDIDO PIÑEIRO

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let \( X \) be a Banach space. We consider sequences \( (x_n) \) in \( X \) lying in the range of a measure valued in a superspace of \( X \) and having bounded variation. Among other results, we prove that G.T. spaces are the only Banach spaces in which those sequences are actually in the range of an \( X^{**} \)-valued measure of bounded variation.

In [PR] it is proved that a Banach space \( X \) is finite-dimensional if and only if every null sequence \( (x_n) \) (equivalently, every compact set) in \( X \) lies inside the range of an \( X \)-valued measure of bounded variation. In view of this result, a natural question arises: Which Banach spaces have the property that every null sequence lies inside the range of a vector measure of bounded variation, if we do not mind the measure is not \( X \)-valued? We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [Ps] and [DU]. We only consider real Banach spaces. If \( X \) is a such space, the phrase "range of an \( X \)-valued measure" always means a set of the form \( \text{rg}(F) = \{F(E) : E \in \Sigma\} \), where \( \Sigma \) is a \( \sigma \)-algebra of subsets of a set \( \Omega \) and \( F : \Sigma \to X \) is countably additive.

Definition 1. Let \( X \) be a Banach space. We say that a subset \( A \) of \( X \) lies in the range of a vector measure of bounded variation (we shorten to a vector bv-measure) provided there exist a Banach space \( X_0 \), an isometry \( J : X \to X_0 \) and a vector measure \( F : \Sigma \to X_0 \) with bounded variation so that \( J(A) \subseteq \text{rg}(F) \).

We are mainly interested in the case \( A \) is countable. In the lemma below we collect some elementary facts.

Lemma 2. Let \( X \) be a Banach space. If \( (x_n) \) is a bounded sequence in \( X \), we consider the linear operator \( T : (\alpha_n) \in l_1 \to \sum \alpha_n x_n \in X \).

Then the following assertions hold:

(i) \( (x_n) \) is in the range of an \( X \)-valued bv-measure iff \( T \) is Pietsch-integral.

(ii) \( (x_n) \) lies inside the range of a vector bv-measure iff \( T \) is 1-summing.

(iii) \( (x_n) \) is in the range of an \( X^{**} \)-valued bv-measure iff \( T \) is integral.
Proof. (i) Although it is well known, we include its proof. Let \( F: \Sigma \to X \) a measure of bounded variation \( \mu \) such that \( \{x_n: n \in \mathbb{N}\} \subset \text{rg}(F) \). Choose \( A_n \in \Sigma \) so that \( F(A_n) = x_n \) for all \( n \in \mathbb{N} \), and define two operators \( P: l_1 \to L^\infty(\mu) \) and \( Q: L^\infty(\mu) \to X \) by

\[
P(\alpha_n) = \sum \alpha_n \chi_{A_n} \quad \text{and} \quad Q(f) = \int f \, dF.
\]

Since \( T = QP \), it suffices to prove that \( Q \) is Pietsch-integral. By \([P1]\) \( Q \) is 1-summing. Then \( Q \) is Pietsch-integral because \( L^\infty(\mu) \) is isometric to a \( \mathcal{E}(K) \)-space \([DU]\). Conversely, if \( T \) is Pietsch-integral, then \( T \) admits a factorization

\[
l_1 \xrightarrow{P} \mathcal{E}(K) \xrightarrow{Q} X
\]

where \( P \) is continuous and \( Q \) is 1-summing \([DU]\). By \([PR, \text{Proposition 1.3}]\) there exists an \( X \)-valued measure \( F \) of bounded variation such that \( \overline{Q(B_{\mathcal{E}(K)})} = \text{rg}(F) \). Hence \( \{x_n: n \in \mathbb{N}\} \subset \text{rg}(\|P\|F) \).

(ii) Suppose \( T \) is 1-summing. If \( J \) denotes the natural isometry into \( l_\infty(B_{X^*}) \), then \( JT \) is Pietsch-integral. By (i) \( (Jx_n) \) lies inside the range of an \( l_\infty(B_{X^*}) \)-valued bv-measure. On the other hand, if \( J: X \to X_o \) is an isometry and \( F: \Sigma \to X_o \) is a bv-measure so that \( \{Jx_n: n \in \mathbb{N}\} \subset \text{rg}(F) \), then (i) tells us that \( JT \) is Pietsch-integral and, therefore, 1-summing. So is \( T \) since \( J \) is an isometry.

(iii) It is obvious since integral and Pietsch-integral operators into \( X^{**} \) are the same \([DU]\).

Remark 3. Recall that a sequence \( (x_n) \) in \( X \) is said to belong to \( l^2_w(X) \) provided we have \( \sum |\langle x_n, x^* \rangle|^2 < +\infty \) for any \( x^* \in X^* \). If \( T: (\alpha_n) \in l_1 \to \sum \alpha_n x_n \in X \) is 1-summing, then \( T \) admits a factorization

\[
l_1 \xrightarrow{P} \mathcal{E}(K) \xrightarrow{Q} X
\]

where \( P \) and \( Q \) are continuous with \( \|P\| \leq 1 \). Hence, there is a sequence \( (z_n) \in l^2_w(X) \) so that

\[
Q(\alpha_n) = \sum \alpha_n z_n \quad \text{for any } (\alpha_n) \in l_2.
\]

Consequently, the class of sequences lying in the range of a bv-measure and the class of sequences which are contained in a weakly 2-summable segment \( \{\sum \alpha_n z_n: (\alpha_n) \in l_2, \|\alpha_n\|_2 \leq 1\} \) are equal.

A particular class of those sequences are the sequences belonging to \( l^2_w(X) \). In \([AD]\) it is proved that such a sequence \( (x_n) \) lies inside the range of the \( X \)-valued measure \( F \) defined by

\[
F(A) = 2 \sum_n \left( \int_A r_n(t) \, dt \right) x_n
\]
for any Lebesgue measurable subset $A$ of $[0, 1]$. But, in general this measure is not of bounded variation. Nevertheless, if $X$ has type 2 and $(x_n) \in l_2^2(X)$ (the Banach space of all sequences $(x_n)$ such that $\|x_n\|_2 = (\sum_k |x_k|^2)^{1/2} < +\infty$), then $F$ has bounded variation. To see this, let $\{A_1, \ldots, A_p\}$ be a finite family of pairwise disjoint Lebesgue measurable subsets of $[0, 1]$. Then
\[
\sum_{i=1}^p \left| \sum_{k=1}^n \left( \int_{A_i} r_k(t) \, dt \right) x_k \right| = \sum_{i=1}^p \left\| \int_{A_i} \left( \sum_{k=1}^n r_k(t) x_k \, dt \right) \right\|
\leq \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| \, dt \leq \left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 \, dt \right)^{1/2} \leq T_2(X) \|x_n\|_2
\]
where $T_2(X)$ is the type 2 constant of $X$. Finally, letting $n \to \infty$ we obtain
\[
\sum_{i=1}^p \|F(A_i)\| \leq 2T_2(X) \|x_n\|_2.
\]
So, $F$ has bounded variation.

If every unconditionally summable sequence $(x_n)$ in $X$ satisfies $\sum_k |x_k|^2 < +\infty$, we say that $X$ has Orlicz property. It is known that cotype 2 spaces have Orlicz property [Ps]. Hence, if $X$ has type 2, $X^*$ has Orlicz property because the dual space $X^*$ has cotype 2 whenever $X$ has type 2. In view of these results, it seems interesting to characterize the Banach spaces $X$ in which every sequence $(x_n) \in l_2^2(X)$ is in the range of an $X$-valued bv-measure. Next Theorem proves that those spaces are the Banach spaces whose dual have Orlicz property.

**Theorem 4.** Let $X$ be a Banach space. The following statements are equivalent:

(i) Every sequence $(x_n) \in l_2^2(X)$ is in the range of an $X$-valued bv-measure.

(ii) Every sequence $(x_n) \in l_2^2(X)$ is in the range of an $X^{**}$-valued bv-measure.

(iii) $X^*$ has Orlicz property.

**Proof.** (i) $\Rightarrow$ (ii) is obvious and (iii) $\Rightarrow$ (i) is straightforward.

(ii) $\Rightarrow$ (iii) The natural inclusion $I: l_2^2(X) \to I(l_1, X)$ is linear and, having closed graph, continuous. Since $(x_n) = \lim_{n \to \infty} (x_1, \ldots, x_n, 0, 0, \ldots)$ for any $(x_n) \in l_2^2(X)$, it follows that $I$ takes $l_2^2(X)$ into $N(l_1, X)$ (note that $N(l_1, X)$ is a subspace of $I(l_1, X)$ because $(l_1)^*$ has the metric approximation property). By transposition we obtain that $B(X, l_1)$ is taken into $l_2^2(X^*)$. So $X^*$ has Orlicz property.

Now we are going to answer our first question.

**Theorem 5.** Let $X$ be a Banach space. The following statements are equivalent:

(i) The unit ball of $X$ lies in the range of a vector bv-measure.

(ii) Every bounded sequence in $X$ is in the range of a vector bv-measure.

(iii) Every null sequence in $X$ is in the range of a vector bv-measure.

(iv) $X$ is isomorphic to a Hilbert space.

**Proof.** Obviously (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv) Again we consider the natural isometry $J: X \to l_\infty(B_{X^{**}})$. By hypothesis, $J$ maps null sequences of $X$ into sequences lying inside the range
of an $l_\infty(B_{X^*})$-valued bv-measure. From [Pñ] it follows that $J^*$ is 1-summing. In particular, $J^*$ is 2-summing, so it admits a factorization

$$\begin{array}{ccc}
l_\infty(B_{X^*})^* & \xrightarrow{J^*} & X^* \\
\downarrow{P} & & \downarrow{Q} \\
L^2(\mu) & & 
\end{array}$$

where $P$ and $Q$ are continuous linear maps. As $J^*$ is a surjection, so is $Q: L^2(\mu) \to X^*$. This proves that $X^*$ is isomorphic to a Hilbert space.

(iv) $\Rightarrow$ (i) If $X$ is isomorphic to a Hilbert space, then the map

$$A: (\xi_x) \in l_1(B_X) \to \sum \xi_x x \in X$$

is 1-summing. Hence, $JA$ is Pietsch-integral. In a similar way to the proof of Lemma 1(i), we can prove that $J(B_X)$ lies inside the range of an $l_\infty(B_{X^*})$-valued bv-measure.

Next, we shall prove that every sequence in $X$, lying in the range of a vector bv-measure, is actually in the range of an $X^{**}$-valued bv-measure if and only if $X^*$ is a G.T. space. Following [Ps] we will say that a Banach space $X$ satisfies Grothendieck’s Theorem (in short G.T.) if

$$B(X, l_2) = \Pi_1(X, l_2).$$

The next theorem shows that G.T. spaces are the only spaces in which every sequence lying in the range of a vector bv-measure actually lies inside the range of an $X^{**}$-valued bv-measure.

**Theorem 6.** Let $X$ be a Banach space. The following are equivalent:

(i) $X^*$ is a G.T. space.

(ii) $\Pi_1(l_1, X) = I(l_1, X)$.

**Proof.** (i) $\Rightarrow$ (ii) Let $T: l_1 \to X$ be 1-summing. Then $T$ admits a factorization:

$$\begin{array}{ccc}
l_1 & \xrightarrow{T} & X \\
\downarrow{P} & & \downarrow{Q} \\
l_2 & & 
\end{array}$$

So $T^* = Q^*P^*$. By hypothesis, $Q^*: X^* \to l_2$ is 1-summing. Hence, $T^*: X^* \to l_\infty$ is integral.

(ii) $\Rightarrow$ (i) Recall that 1-summing and 2-summing operators on $l_1$ are the same because $l_1$ has cotype 2 [Ps]. Consequently, by the open mapping theorem there is a positive constant $c$ such that

$$i(T) \leq c\pi_2(T) \quad \text{for any } T \in \Pi_2(l_1, X).$$

Then, we have

$$i(T) \leq c\pi_2(T) \quad \text{for any } T \in \Pi_2(l_n^1, X).$$

But the integral norm and the nuclear norm of an operator defined on a finite-dimensional space are equal. So, we can write (1) in the form

$$\nu(T) \leq c\pi_2(T) \quad \text{for any } n \text{ and for any } T: l_n^1 \to X.$$
Now, by the duality argument [T, p. 18], \((1')\) yields

\[ \pi_2(T) \leq c\|T\| \text{ for any } n \text{ and for any } T: X \to l^n. \]

The result follows from [Ps, Theorem 6.2].

**Final notes and examples**

1. It is well known that the unit ball of \(l_2\) is the range of and \(l_2\)-valued countably additive measure [AD]. Therefore, so is every weakly 2-summable segment \( \{\sum \alpha_n z_n : \|(\alpha_n)\|_2 \leq 1\} \). Then Remark 3 shows that every sequence lying in the range of a vector bv-measure is actually in the range of an \(X\)-valued countably additive measure. Nevertheless, the converse result is not true. To see this, take the space \(l_\infty\). As mentioned earlier, there are null sequences in \(l_\infty\) which do not lie in the range of an \(l_\infty\)-valued bv-measure but they do in the range of an \(l_\infty\)-valued measure since \((l_\infty)^*\) is an \(L^1(\mu)\)-space [PR].

2. We know from Lemma 2 that a sequence \((x_n)\) in \(X\) lies in the range of an \(X^{**}\)-valued bv-measure iff \(T: (\alpha_n) \in l_1 \to \sum \alpha_n x_n \in X\) is integral, equivalently, iff \(T^*: x^* \in X^* \to \langle (x_n, x^*) \rangle_n \in l_\infty\) is 1-summing. We have obtained the following characterization of these sequences:

A sequence \((x_n)\) in \(X\) lies in the range of an \(X^{**}\)-valued bv-measure iff there is a positive constant \(c\) such that

\[ \sum_n |\langle x_n, \phi(n) \rangle| \leq c \sup \left\{ \sum_n |\langle \phi(n), x^{**} \rangle| : \|x^{**}\| \leq 1 \right\} \]

for any \(\phi: N \to X^*\) so that the set \(\{n \in N : \phi(n) \neq 0\}\) is finite.

Necessity is obvious. To prove that \(T^*\) is 1-summing if (2) holds, we apply Ky Fan’s Lemma [P2] to the collection \(\mathcal{F}\) of all functions \(\Phi\) defined by

\[ \Phi(\mu) = \sum_n (|\langle x_n, \phi(n) \rangle| - c \int_{B_{X^{**}}} |\langle \phi(n), x^{**} \rangle| d\mu), \]

where \(\phi: N \to X^*\) is as before and \(\mu\) is a probability measure on the unit ball of \(X^{**}\) endowed with the topology \(\sigma(X^{**}, X^*)/B_{X^{**}}\). If \(W(B_{X^{**}})\) denotes the set of all such probability measures, then every \(\Phi \in \mathcal{F}\) is convex and continuous on \(W(B_{X^{**}})\) endowed with the \(w^*\)-topology. Using Dirac measures, we can prove that, given \(\Phi \in \mathcal{F}\), there is \(\mu \in W(B_{X^{**}})\) so that \(\Phi(\mu) \leq 0\). Since \(\mathcal{F}\) is concave, it follows that there is \(\mu \in W(B_{X^{**}})\) satisfying \(\Phi(\mu) \leq 0\) for any \(\Phi \in \mathcal{F}\). In particular, it follows that

\[ |\langle x_n, x^* \rangle| \leq c \int_{B_{X^{**}}} |\langle x^*, x^{**} \rangle| d\mu \]

for any \(n \in N\) and any \(x^* \in X^*\). This proves that \(T^*\) is 1-summing.

3. Integral and Pietsch-integral operators into \(L^1(\mu)\) are the same [DU, p. 259]. Then, a sequence \((f_n)\) in \(L^1(\mu)\) lies in the range of an \(L^1(\mu)\)-valued bv-measure iff there is \(g \in L^1(\mu)\) such that \(|f_n| \leq g\) \(\mu\)-almost everywhere for any \(n \in N\). To prove this, it suffices to recall Grothendieck’s Theorem about integral operators into \(L^1(\mu)\): \(T: X \to L^1(\mu)\) is integral if and only if \(T(B_X)\) is a lattice bounded subset of \(L^1(\mu)\).
CÁNIDO PIÑEIRO

REFERENCES


DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SEVILLA, Aptdo. 1160, SEVILLA, 41080, SPAIN

Current address: Departamento de Matemáticas, Escuela Politécnica Superior, Universidad de Huelva, La Rábida, Huelva, Spain

E-mail address: candido@colon.uhu.es