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SEQUENCES IN THE RANGE OF A VECTOR MEASURE WITH BOUNDED VARIATION

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ABSTRACT. Let X be a Banach space. We consider sequences (x_n) in X lying in the range of a measure valued in a superspace of X and having bounded variation. Among other results, we prove that G.T. spaces are the only Banach spaces in which those sequences are actually in the range of an X^{**}-valued measure of bounded variation.

In [PR] it is proved that a Banach space X is finite-dimensional if and only if every null sequence (x_n) (equivalently, every compact set) in X lies inside the range of an X-valued measure of bounded variation. In view of this result, a natural question arises: Which Banach spaces have the property that every null sequence lies inside the range of a vector measure of bounded variation, if we do not mind the measure is not X-valued? We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [Ps] and [DU]. We only consider real Banach spaces. If X is a such space, the phrase "range of an X-valued measure" always means a set of the form $\operatorname{rg}(F) = \{F(E) : E \in \Sigma\}$, where Σ is a σ-algebra of subsets of a set Ω and $F : \Sigma \to X$ is countably additive.

Definition 1. Let X be a Banach space. We say that a subset A of X lies in the range of a vector measure of bounded variation (we shorten to a vector bv-measure) provided there exist a Banach space X_0, an isometry $J : X \to X_0$ and a vector measure $F : \Sigma \to X_0$ with bounded variation so that $J(A) \subset \operatorname{rg}(F)$.

We are mainly interested in the case A is countable. In the lemma below we collect some elementary facts.

Lemma 2. *Let X be a Banach space. If (x_n) is a bounded sequence in X, we consider the linear operator $T : (\alpha_n) \in l_1 \to \sum \alpha_n x_n \in X$.*
Then the following assertions hold:

(i) *(x_n) is in the range of an X-valued bv-measure iff T is Pietsch-integral.*
(ii) *(x_n) lies inside the range of a vector bv-measure iff T is 1-summing.*
(iii) *(x_n) is in the range of an X^{**}-valued bv-measure iff T is integral.*

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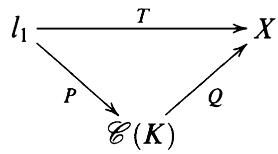
Proof. (i) Although it is well known, we include its proof. Let $F: \Sigma \to X$ a measure of bounded variation μ such that $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg}(F)$. Choose $A_n \in \Sigma$ so that $F(A_n) = x_n$ for all $n \in \mathbb{N}$, and define two operators $P: l_1 \to L^\infty(\mu)$ and $Q: L^\infty(\mu) \to X$ by

$$P(\alpha_n) = \sum \alpha_n \chi_{A_n} \quad \text{and} \quad Q(f) = \int f\, dF.$$

Since $T = QP$, it suffices to prove that Q is Pietsch-integral. By [P1] Q is 1-summing. Then Q is Pietsch-integral because $L^\infty(\mu)$ is isometric to a $\mathscr{C}(K)$-space [DU]. Conversely, if T is Pietsch-integral, then T admits a factorization

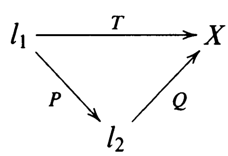


where P is continuous and Q is 1-summing [DU]. By [PR, Proposition 1.3] there exists an X-valued measure F of bounded variation such that $\overline{Q(B_{\mathscr{C}(K)})} = \operatorname{rg}(F)$. Hence $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg}(\|P\| F)$.

(ii) Suppose T is 1-summing. If J denotes the natural isometry into $l_\infty(B_{X^*})$, then JT is Pietsch-integral. By (i) (Jx_n) lies inside the range of an $l_\infty(B_{X^*})$-valued bv-measure. On the other hand, if $J: X \to X_o$ is an isometry and $F: \Sigma \to X_o$ is a bv-measure so that $\{Jx_n : n \in \mathbb{N}\} \subseteq \operatorname{rg}(F)$, then (i) tells us that JT is Pietsch-integral and, therefore, 1-summing. So is T since J is an isometry.

(iii) It is obvious since integral and Pietsch-integral operators into X^{**} are the same [DU].

Remark 3. Recall that a sequence (x_n) in X is said to belong to $l_w^2(X)$ provided we have $\sum |\langle x_n, x^* \rangle|^2 < +\infty$ for any $x^* \in X^*$. If $T: (\alpha_n) \in l_1 \to \sum \alpha_n x_n \in X$ is 1-summing, then T admits a factorization



where P and Q are continuous with $\|P\| \le 1$. Hence, there is a sequence $(z_n) \in l_w^2(X)$ so that

$$Q(\alpha_n) = \sum \alpha_n z_n \quad \text{for any } (\alpha_n) \in l_2.$$

Consequently, the class of sequences lying in the range of a bv-measure and the class of sequences which are contained in a weakly 2-summable segment $\{\sum \alpha_n z_n : (\alpha_n) \in l_2,\ \|(\alpha_n)\|_2 \le 1\}$ are equal.

A particular class of those sequences are the sequences belonging to $l_w^2(X)$. In [AD] it is proved that such a sequence (x_n) lies inside the range of the X-valued measure F defined by

$$F(A) = 2 \sum_n \left(\int_A r_n(t)\, dt \right) x_n$$

for any Lebesgue measurable subset A of $[0, 1]$. But, in general this measure is not of bounded variation. Nevertheless, if X has type 2 and $(x_n) \in l_a^2(X)$ (the Banach space of all sequences (x_n) such that $\|(x_n)\|_2 = (\sum \|x_n\|^2)^{1/2} < +\infty$), then F has bounded variation. To see this, let $\{A_1, \dots, A_p\}$ be a finite family of pairwise disjoint Lebesgue measurable subsets of $[0, 1]$. Then

$$\sum_{i=1}^{p} \left\| \sum_{k=1}^{n} \left(\int_{A_i} r_k(t)dt \right) x_k \right\| = \sum_{i=1}^{p} \left\| \int_{A_i} \left(\sum_{k=1}^{n} r_k(t) x_k \, dt \right) \right\|$$

$$\leq \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\| dt \leq \left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|^2 dt \right)^{1/2} \leq T_2(X) \|(x_n)\|_2$$

where $T_2(X)$ is the type 2 constant of X. Finally, letting $n \to \infty$ we obtain

$$\sum_{i=1}^{p} \|F(A_i)\| \leq 2T_2(X)\|(x_n)\|_2.$$

So, F has bounded variation.

If every unconditionally summable sequence (x_n) in X satisfies $\sum \|x_n\|^2 < +\infty$, we say that X has Orlicz property. It is known that cotype 2 spaces have Orlicz property [Ps]. Hence, if X has type 2, X^* has Orlicz property because the dual space X^* has cotype 2 whenever X has type 2. In view of these results, it seems interesting to characterize the Banach spaces X in which every sequence $(x_n) \in l_a^2(X)$ is in the range of an X-valued bv-measure. Next Theorem proves that those spaces are the Banach spaces whose dual have Orlicz property.

Theorem 4. *Let X be a Banach space. The following statements are equivalent:*

(i) *Every sequence $(x_n) \in l_a^2(X)$ is in the range of an X-valued bv-measure.*
(ii) *Every sequence $(x_n) \in l_a^2(X)$ is in the range of an X^{**}-valued bv-measure.*
(iii) *X^* has Orlicz property.*

Proof. (i) $\Rightarrow$ (ii) is obvious and (iii) $\Rightarrow$ (i) is straightforward.

(ii) $\Rightarrow$ (iii) The natural inclusion $I\colon l_a^2(X) \to I(l_1, X)$ is linear and, having closed graph, continuous. Since $(x_n) = \lim_{n\to\infty}(x_1, \dots, x_n, 0, 0, \dots)$ for any $(x_n) \in l_a^2(X)$, it follows that I takes $l_a^2(X)$ into $N(l_1, X)$ (note that $N(l_1, X)$ is a subspace of $I(l_1, X)$ because $(l_1)^*$ has the metric approximation property). By transposition we obtain that $B(X, l_1)$ is taken into $l_a^2(X^*)$. So X^* has Orlicz property.

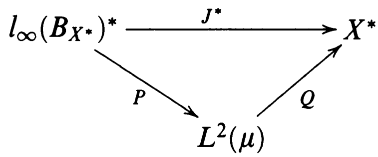
Now we are going to answer our first question.

Theorem 5. *Let X be a Banach space. The following statements are equivalent:*

(i) *The unit ball of X lies in the range of a vector bv-measure.*
(ii) *Every bounded sequence in X is in the range of a vector bv-measure.*
(iii) *Every null sequence in X is in the range of a vector bv-measure.*
(iv) *X is isomorphic to a Hilbert space.*

Proof. Obviously (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv) Again we consider the natural isometry $J\colon X \to l_\infty(B_{X^*})$. By hypothesis, J maps null sequences of X into sequences lying inside the range of an $l_\infty(B_{X^*})$-valued bv-measure. From [Pñ] it follows that J^* is 1-summing. In particular, J^* is 2-summing, so it admits a factorization



where P and Q are continuous linear maps. As J^* is a surjection, so is $Q: L^2(\mu) \to X^*$. This proves that X^* is isomorphic to a Hilbert space.

(iv) $\Rightarrow$ (i) If X is isomorphic to a Hilbert space, then the map

$$A: (\xi_x) \in l_1(B_X) \to \sum \xi_x x \in X$$

is 1-summing. Hence, JA is Pietsch-integral. In a similar way to the proof of Lemma 1(i), we can prove that $J(B_X)$ lies inside the range of an $l_\infty(B_{X^*})$-valued bv-measure.

Next, we shall prove that every sequence in X, lying in the range of a vector bv-measure, is actually in the range of an X^{**}-valued bv-measure if and only if X^* is a G.T. space. Following [Ps] we will say that a Banach space X satisfies Grothendieck's Theorem (in short G.T.) if

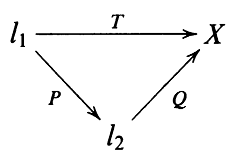
$$B(X, l_2) = \Pi_1(X, l_2).$$

The next theorem shows that G.T. spaces are the only spaces in which every sequence lying in the range of a vector bv-measure actually lies inside the range of an X^{**}-valued bv-measure.

Theorem 6. *Let X be a Banach space. The following are equivalent*:

(i) X^* *is a G.T. space.*
(ii) $\Pi_1(l_1, X) = I(l_1, X)$.

Proof. (i) $\Rightarrow$ (ii) Let $T: l_1 \to X$ be 1-summing. Then T admits a factorization:



So $T^* = Q^*P^*$. By hypothesis, $Q^*: X^* \to l_2$ is 1-summing. Hence, $T^*: X^* \to l_\infty$ is integral.

(ii) $\Rightarrow$ (i) Recall that 1-summing and 2-summing operators on l_1 are the same because l_1 has cotype 2 [Ps]. Consequently, by the open mapping theorem there is a positive constant c such that

$$i(T) \le c\pi_2(T) \quad \text{for any } T \in \Pi_2(l_1, X).$$

Then, we have

$$(1) \qquad i(T) \le c\pi_2(T) \quad \text{for any } n \text{ and any } T \in \Pi_2(l_1^n, X).$$

But the integral norm and the nuclear norm of an operator defined on a finite-dimensional space are equal. So, we can write (1) in the form

$$(1') \qquad \nu(T) \le c\pi_2(T) \quad \text{for any } n \text{ and for any } T: l_1^n \to X.$$

Now, by the duality argument [T, p. 18], $(1')$ yields

$$\pi_2(T) \leq c\|T\| \quad \text{for any } n \text{ and for any } T\colon X \to l_1^n.$$

The result follows from [Ps, Theorem 6.2].

Final notes and examples

1. It is well known that the unit ball of l_2 is the range of and l_2-valued countably additive measure [AD]. Therefore, so is every weakly 2-summable segment $\{\sum \alpha_n z_n\colon \|(\alpha_n)\|_2 \leq 1\}$. Then Remark 3 shows that every sequence lying in the range of a vector bv-measure is actually in the range of an X-valued countably additive measure. Nevertheless, the converse result is not true. To see this, take the space l_∞. As mentioned earlier, there are null sequences in l_∞ which do not lie in the range of an l_∞-valued bv-measure but they do in the range of an l_∞-valued measure since $(l_\infty)^*$ is an $L^1(\mu)$-space [PR].

2. We know from Lemma 2 that a sequence (x_n) in X lies in the range of an X^{**}-valued bv-measure iff $T\colon (\alpha_n) \in l_1 \to \sum \alpha_n x_n \in X$ is integral, equivalently, iff $T^*\colon x^* \in X^* \to (\langle x_n, x^*\rangle)_n \in l_\infty$ is 1-summing. We have obtained the following characterization of these sequences:

A sequence (x_n) in X lies in the range of an X^{**}-valued bv-measure iff there is a positive constant c such that

$$\sum_n |\langle x_n, \phi(n)\rangle| \leq c \sup\left\{\sum_n |\langle \phi(n), x^{**}\rangle|\colon \|x^{**}\| \leq 1\right\} \tag{2}$$

for any $\phi\colon \mathbb{N} \to X^*$ so that the set $\{n \in \mathbb{N}\colon \phi(n) \neq 0\}$ is finite.

Necessity is obvious. To prove that T^* is 1-summing if (2) holds, we apply Ky Fan's Lemma [P2] to the collection $\mathscr{F}$ of all functions Φ defined by

$$\Phi(\mu) = \sum_n (|\langle x_n, \phi(n)\rangle| - c\int_{B_{X^{**}}} |\langle \phi(n), x^{**}\rangle| d\mu),$$

where $\phi\colon \mathbb{N} \to X^*$ is as before and μ is a probability measure on the unit ball of X^{**} endowed with the topology $\sigma(X^{**}, X^*)/_{B_{X^{**}}}$. If $W(B_{X^{**}})$ denotes the set of all such probability measures, then every $\Phi \in \mathscr{F}$ is convex and continuous on $W(B_{X^{**}})$ endowed with the w*-topology. Using Dirac measures, we can prove that, given $\Phi \in \mathscr{F}$, there is $\mu \in W(B_{X^{**}})$ so that $\Phi(\mu) \leq 0$. Since $\mathscr{F}$ is concave, it follows that there is $\mu \in W(B_{X^{**}})$ satisfying $\Phi(\mu) \leq 0$ for any $\Phi \in \mathscr{F}$. In particular, it follows that

$$|\langle x_n, x^*\rangle| \leq c\int_{B_{X^{**}}} |\langle x^*, x^{**}\rangle| d\mu$$

for any $n \in \mathbb{N}$ and any $x^* \in X^*$. This proves that T^* is 1-summing.

3. Integral and Pietsch-integral operators into $L^1(\mu)$ are the same [DU, p. 259]. Then, a sequence (f_n) in $L^1(\mu)$ lies in the range of an $L^1(\mu)$-valued bv-measure iff there is $g \in L^1(\mu)$ such that $|f_n| \leq g$ μ-almost everywhere for any $n \in \mathbb{N}$. To prove this, it suffices to recall Grothendieck's Theorem about integral operators into $L^1(\mu)$: *$T\colon X \to L^1(\mu)$ is integral if and only if $T(B_X)$ is a lattice bounded subset of $L^1(\mu)$.*

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