ANALYTIC ULTRADISTRIBUTIONS

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Abstract. A necessary and sufficient condition that an ultradistribution, of Beurling or Roumieu type, which is defined on an open set \( \Omega \subset \mathbb{R}^n \) is a real analytic function is given. This result is applied to different problems.

1. Introduction

It is of interest to know whether a generalized function \( T \) or its restriction to an open set is defined by a real analytic function, especially if this generalized function is a solution of an equation. One can find answers for different classes of generalized functions in [2], [3] and [5]. An answer on this question for distributions was made by L. Schwartz (see [8, Chapter VI, Theorem XXIV]). Our aim is to extend this result to ultradistributions. The key of the proof of the mentioned theorem in [8] was in the parametrix. Therefore we cannot proceed analogously to answer this question for ultradistributions. In our proof we use an assertion which can be found in Komatsu [4] and applied also by Pilipović [7]. We cite this assertion as Theorem A.

As an illustration of applications of Theorem 1 we give two direct consequences and two theorems concerning convolution equations and Painlevé’s theorem.

2. Notation

We follow the notation of [6]. Let us repeat some.

By \( M_p \) we denote a sequence of positive numbers satisfying some of the following conditions: \( M_0 = M_1 = 1 \) and

\[
\begin{align*}
M_p^2 & \leq M_{p-1}M_{p+1}, \ p \in \mathcal{N}; \\
M_p/(M_qM_{p-q}) & \leq AB^p, \ 0 \leq q \leq p, \ p, \ q \in \mathcal{N}; \\
\sum_{q=p+1}^\infty M_{q-1}/M_q & \leq ApM_p/M_{p+1}, \ p \in \mathcal{N},
\end{align*}
\]

where \( A \) and \( B \) are constants independent on \( p \).

We use two classes of ultradifferentiable functions, the Beurling class and Reumieu class (in short, \( (M_p) \) class and \( \{M_p\} \) class).

Let \( u \) be a positive number and \( \{u_p\} \) be a sequence of positive numbers...
increasing to $\infty$. We denote

\begin{equation}
H_p = \begin{cases}
u^p, & \text{for the class } (M_p) \\ u_1 \cdots u_p, & \text{for the class } \{M_p\} \end{cases}.
\end{equation}

Let $\Omega$ be an open set in $\mathbb{R}^n$.

By $\mathcal{D}^{H_p}_{K}$ is denoted the Banach space of all $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ with support in $K$ such that

$$\sup_{x \in K} |f^{(p)}(x)|/H_p \to 0 \quad \text{as } p \to \infty$$

with the norm

\begin{equation}
q_{H_p,M_p}(f) = \sup_{x \in K, p \in \mathcal{M}^n_0} |f^{(p)}(x)|/H_p M_p,
\end{equation}

where $K$ is a compact set in $\mathbb{R}^n$. Then the basic spaces are

$$\mathcal{D}^{(M_p)}_{K} = \text{proj lim } \mathcal{D}^{H_p}_{K}, \quad \mathcal{D}^{\{M_p\}}_{K} = \text{proj lim } H_p^{H_p}_{K}$$

and

$$\mathcal{D}^*(\Omega) = \text{ind lim } \mathcal{D}^*_K,$$

where $*$ denotes either $(M_p)$ or $\{M_p\}$.

An operator of the form

$$P(D) = \sum_{|i| \geq 0} a_i D^i, \quad a_i \in \mathbb{C},$$

is called the ultradifferential operator of class $(M_p)$ (of class $\{M_p\}$) if there are constants $L$ and $C$ (for every $L > 0$ there is a constant $C$) such that

$$|a_i| \leq CL^{|i|}/M_i, \quad i \in \mathcal{M}^n_0.$$

$A(\Omega)$ is the space of real analytic functions.

$B_r = B(0, r)$ is the open ball with center at zero and with radius $r$.

**Theorem A** (see [4, Theorem 2.11]). Let the sequence $M_p$ satisfy conditions $(M.1)$, $(M.2)$ and $(M.3)$. For a given $H_p$ from (1) and a compact neighbourhood $Q$ of zero in $\mathbb{R}^n$ there exist an ultradifferential operator $P(D)$ of class $\ast$ and two functions $\varphi \in \mathcal{C}^\infty$ and $w \in D^*_Q$ such that

$$P(D)\varphi = \delta + w, \quad \sup \varphi \subset Q$$

and

$$\sup_{x \in Q} |\varphi^{(i)}(x)|/H_i M_i \to 0, \quad |i| \to \infty.$$

3. **Main result**

**Theorem 1.** Suppose that $M_p$ satisfies $(M.1)$, $(M.2)$ and $(M.3)$ and that $\Omega, \Omega_1$ are two open sets in $\mathbb{R}^n$ such that $\Omega = \Omega_1 - B_r$ for some $r > 0$. An ultradistribution $T \in \mathcal{D}'(\Omega)$ is defined by the real analytic function $f$, $f \in \mathcal{S}(\Omega)$, if and only if $T \ast w \in \mathcal{S}(\Omega_1)$ for every $w \in \mathcal{D}^*_K$, where $K = B_r$ and $\ast$ is the sign of convolution.

**Proof.** If $T = f \in \mathcal{A}(\Omega)$, then it can be characterized as an infinitely differentiable function such that for every compact set $K' \subset \Omega$ and $p \in \mathcal{M}_0^n$ there exist two constants $C_{K'}$ and $C$ for which

$$\sup_{x \in K'} |f^{(p)}(x)| \leq Cp!C_{K'}^{|p|}, \quad p \in \mathcal{M}_0^n.$$
Take any compact set $K'' \subset \Omega_1$. Then
\[
\sup_{x \in K''} |(f * w)^{(p)}(x)| \leq C_p C_{K''}^{[p]} \int_{\mathbb{R}^n} |w(x)| \, dx, \quad p \in \mathcal{N}_0^n,
\]
which proves that $f * w \in \mathcal{A}(\Omega_1)$.

Suppose now that for $T \in \mathcal{D}'(\Omega)$ and for every $w \in \mathcal{D}_K^*$, $T * w \in \mathcal{A}(\Omega_1)$ and $A$ is an open and relatively compact set, $A \subset \Omega \subset \Omega_1$. In the first step of the proof we shall analyse the functional $R$,
\[
R: (w, q) \mapsto \sup_{x \in A} |(f * w)^{(q)}(x)|/q!, \quad w \in \mathcal{D}_K^*, \; q \in \mathcal{N}^n,
\]
which is related to the convergence radii of the power series
\[
\sum_{|q| \geq 0} (f * w)^{(q)}(x)(x - y)^q/q!, \quad x \in A.
\]

For a fixed $q \in \mathcal{N}^n$, $R$ is continuous in $w \in \mathcal{D}_K^*$, and for a fixed $w \in \mathcal{D}_K^*$, $R$ is bounded in $q \in \mathcal{N}^n$. Since $\mathcal{D}_K^*$ is a barreled space, by the Banach theorem it follows that the family $\{R(\cdot, q), q \in \mathcal{N}^n\}$ is equicontinuous. Then for a fixed $L > 0$ there exist $\beta > 0$ and $H_p$ such that $R(w, q) < L$ when $w \in V_\beta$ and $q \in \mathcal{N}^n$, where $V_\beta$ is the neighbourhood of zero in $\mathcal{D}_K^*$ of the form
\[
V_\beta = \left\{ \phi \in \mathcal{D}_K^*: \sup_{x \in \mathbb{R}^n, p \in \mathcal{N}_0^p} |\phi^{(p)}(x)|/H_p M_p < \beta \right\}.
\]

From the properties of the functional $R$ it follows that the family of functions
\[
F_q: w \mapsto (T * w)^{(q)}/q! L^{q|q|} = (D^q T * w)/q! L^{q|q|}, \quad q \in \mathcal{N}_0^n,
\]
is equicontinuous; $F_q$ maps $\mathcal{D}_K^*$ into $(\mathcal{B})_A$. $(\mathcal{B})_A$ is the space of continuous and bounded functions on $A$. Also, for every $w \in V_\beta$ and for every $q \in \mathcal{N}_0^n$, $(T * w)^{(q)}/q! L^{q|q|}$ belongs to the ball $B(0, L) \subset (\mathcal{B})_A$. Namely, there exists $C > 0$ such that
\[
|(T * w)^{(q)}(x)| \leq C q! L^{q|q|}, \quad x \in \overline{A}, \; w \in V_\beta.
\]

Denote by $\mathcal{D}_K^{H_p M_p}$ the completion of $\mathcal{D}_K^*$ under the norm $q_{H_p M_p}$ given by (2), where $H_p$ is fixed by $V_\beta$. In the second part of the proof we shall show that the family $\{F_q: q \in \mathcal{N}_0^n\}$ can be extended by $\mathcal{D}_K^*$ to $\mathcal{D}_K^{H_p M_p}$, keeping uniform continuity; let us denote this extension by $\{\overline{F_q}: q \in \mathcal{N}_0^n\}$.

For this purpose we use the theorem on the extension of a function by continuity (see [1, Chapter I, §8.5]). Let $w_0 \in \mathcal{D}_K^{H_p M_p}$ and let $\{w_j\} \subset \mathcal{D}_K^*$ be the sequence which converges to $w_0$ in $\mathcal{D}_K^{H_p M_p}$. We shall prove that $F_q(w_j)$ converges in $(\mathcal{B})_A$ when $j \to \infty$, uniformly in $q \in \mathcal{N}_0^n$. To do this we shall show that $\{F_q(w_j), j \in \mathcal{N}_0^n\}$ is a Cauchy sequence in $(\mathcal{B})_A$, uniform in $q \in \mathcal{N}_0^n$.

Let $W$ be a neighbourhood of zero in $(\mathcal{B})_A$. Then it contains the ball $B(0, L) \subset (\mathcal{B})_A$ for some $L > 0$. Consequently, $F_q(V_\beta) \subset W, q \in \mathcal{N}_0^n$, where $V_\beta$ is given by (3). Let $j_0$ be such that $w_i - w_j \in V_\beta, i, j \geq j_0$. Then
\[
F_q(w_i) - F_q(w_j) = F_q(w_i - w_j) \in W, \quad i, j \geq j_0, q \in \mathcal{N}_0^n,
\]
and \( F_q(w_j) \) converges to \( h_q \) in \((\mathcal{EB})_A\), when \( j \to \infty \), uniformly in \( q \in \mathcal{N}_0^n \).

By the cited extension theorem it follows that \( F_q(w) = h_q \in (\mathcal{EB})_A, \ q \in \mathcal{N}_0^n \); every \( F_q, \ q \in \mathcal{N}_0^n \), is uniquely defined.

We shall prove that

\[
\begin{align*}
\text{for } q \in \mathcal{N}_0^n: \\
 h_q = D^q(T * \tilde{w})/q! L^{[q]} = (D^q T * \tilde{w})/q! L^{[q]},
\end{align*}
\]

The sequence \( \{w_j\} \) converges to \( \tilde{w} \) in \( \mathcal{E}'_B^* \), as well. Therefore \( (T * w_j)^{(q)} = D^q T * w_j \) converges to \( D^q T * \tilde{w} \) in \( \mathcal{D}'_A^* \), when \( j \to \infty, \ q \in \mathcal{N}_0^n \) (see [6, Theorem 6.12]). Thus \( (D^q T * \tilde{w})/q! L^{[q]} \) must be \( h_q, q \in \mathcal{N}_0^n \). Since the derivative is a continuous mapping of \( \mathcal{D}'_A^* \) into \( \mathcal{D}'_A^* \), it follows that \( h_q = D^q(T * \tilde{w})/q! L^{[q]} \), as well. Consequently, \( T * \tilde{w} \in \mathcal{A}(\Omega_1) \).

In the third step of the proof it remains only to use Theorem A. Let us remark that if \( \varphi \in C^\infty \) has support in the compact set \( Q \subset K \) and \( qH_p, q \in \mathcal{N}_0^n \), then \( \varphi \in \mathcal{D}'_K H_p \). We also know that \( \mathcal{A}(\omega) \subset \mathcal{E}'(\omega) \) and that \( P(D) \mathcal{E}'(\omega) \subset \mathcal{E}'(\omega) \) for any open set \( \omega \subset \mathcal{R}^n \). According to the above remark we deduce from Theorem A that

\[
D^q T/q! L^{[q]} = P(D)(\varphi * D^q T)/q! L^{[q]} + w * D^q T \quad \text{on } \Omega, \ q \in \mathcal{N}_0^n.
\]

Therefore \( T \in \mathcal{A}(\Omega) \).

4. Applications of Theorem 1

Direct consequences of Theorem 1 are:

1. Lemma 2.4 in [4] we know the analytic form of the operator \( P(D) \) and of the functions \( \varphi \) and \( w \) given in Theorem A. With these \( P(D), \ \varphi \) and \( w \), Theorem 1 asserts that the equation \( (P(D)X)(x) + (w * f)(x) = f(x), \quad x \in \Omega \), has a solution \( X = (\varphi * f) \in \mathcal{A}(\Omega) \) for any \( f \in \mathcal{A}(\Omega) \).

2. Denote by \( \delta_h \) the distribution \( \delta \) shifted in the point \( h \). The function \( H: h \to \delta_h * T \) maps \( \mathcal{R}^n \) into \( \mathcal{D}'(\mathcal{R}^n) \) and has all derivatives. Theorem 1 asserts that \( H \) is real analytic if and only if \( T \) is defined by a real analytic function. The property that \( H \) is real analytic means that the set \( \{D^q(\delta_h * T)/q! L^{[q]}, \ q \in \mathcal{N}_0^n, \ h \in K \} \) is bounded in \( D'(\mathcal{R}^n) \) for every compact set \( K \subset \mathcal{R}^n \) and an \( L > 0 \) which depends on \( K \), namely that

\[
\sup_{h \in K, \ q \in \mathcal{N}_0^n} |D^q(T * w)(h)|/q! L^{[q]} < C \quad \text{for every } w \in \mathcal{D}'(\mathcal{R}^n)
\]

where \( C > 0 \) depends on \( K \) and \( w \in \mathcal{D}'(\mathcal{R}^n) \).

Application of Theorem 1 to convolution equations. Let

\[
A * T = \sum_{k=1}^m A_{j,k} * T_k, \quad j = 1, \ldots, m,
\]

where \( A = (A_{j,k}) \) is a given \((m \times m)\)-matrix of elements belonging to \( \mathcal{E}'(\mathcal{R}^n) \) and \( T \) is an \( m \)-tuple \( \{T_1, \ldots, T_m\} \) of unknown ultradistributions.

Theorem 2. Suppose that the system \( T * A = 0 \) has the following property: Any solution which belongs to \( (\mathcal{E}'(\Omega_1))^m \) belongs to \( (\mathcal{A}(\Omega_1))^m \) as well. Then for
every \( m \)-tuple \( U \) of ultradistributions which is a solution of the system \( T \ast A = 0 \) there exists \( r > 0 \) such that \( U \in (\mathcal{A}(\Omega))^m, \Omega = \Omega_1 - B_r \).

Proof. For the sake of simplicity we shall prove Theorem 2 only in case \( m = 1 \). Suppose that \( U \in \mathcal{D}^*(\Omega) \) is a solution to equation \( T \ast A = 0 \). Then \( U \ast w \in \mathcal{E}^*(\Omega_1) \) for every \( w \in \mathcal{D}^*(B_r) \) satisfies equation \( T \ast A = 0 \) as well, because of properties of the convolution. Since \( U \ast w \in \mathcal{E}^*(\Omega_1) \), by the supposition in Theorem 2, \( U \ast w \) belongs to \( \mathcal{A}(\Omega_1) \). Theorem 1 asserts that \( U \in \mathcal{A}(\Omega) \). This is the end of the proof.

Application of Theorem 1 to Painlevé's theorem. Let \( V \) be an open set in \( \mathcal{E} \) and \( \Omega = V \cap \mathcal{R} \). Classical Painlevé's theorem asserts that a function \( f \) holomorphic on \( V \setminus \Omega \) and continuous on \( V \) is holomorphic on the whole set \( V \). It is well known that the continuity can be replaced by the existence and equality of the limits \( \lim_{y \to 0^+} f(x \pm iy) \) in \( \mathcal{D}'(\Omega) \). Theorem 1 admits to weaken this condition, supposing that the above limits exist in \( \mathcal{D}^*(\Omega) \).

**Theorem 3.** Let \( f \) be a holomorphic function on \( V \setminus \Omega \). If the limits \( f(x \pm i0) = \lim_{y \to 0^+} f(x \pm iy) \) exist in \( \mathcal{D}^*(\Omega) \) and \( f(x + i0) = f(x - i0) \), then \( f \) is holomorphic on \( V \).

**Proof.** The method of the proof is just the same as for distributions. First we have to apply classical Painlevé's theorem to the convolution \( f \ast w, w \in \mathcal{D}_K^* \), and then use Theorem 1.

**References**


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