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# ANALYTIC ULTRADISTRIBUTIONS

B. STANKOVIĆ

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ABSTRACT. A necessary and sufficient condition that an ultradistribution, of Beurling or Roumieu type, which is defined on an open set $\Omega \subset \mathscr{R}^n$ is a real analytic function is given. This result is applied to different problems.

## 1. INTRODUCTION

It is of interest to know whether a generalized function $T$ or its restriction to an open set is defined by a real analytic function, especially if this generalized function is a solution of an equation. One can find answers for different classes of generalized functions in [2], [3] and [5]. An answer on this question for distributions was made by L. Schwartz (see [8, Chapter VI, Theorem XXIV]). Our aim is to extend this result to ultradistributions. The key of the proof of the mentioned theorem in [8] was in the parametrix. Therefore we cannot proceed analogously to answer this question for ultradistributions. In our proof we use an assertion which can be found in Komatsu [4] and applied also by Pilipović [7]. We cite this assertion as Theorem A.

As an illustration of applications of Theorem 1 we give two direct consequences and two theorems concerning convolution equations and Painlevé's theorem.

## 2. NOTATION

We follow the notation of [6]. Let us repeat some.

By $M_p$ we denote a sequence of positive numbers satisfying some of the following conditions: $M_0 = M_1 = 1$ and

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathscr{N}$;

(M.2) $M_p/(M_qM_{p-q}) \leq AB^p$, $0 \leq q \leq p$, $p, q \in \mathscr{N}$;

(M.3) $\sum_{q=p+1}^{\infty} M_{q-1}/M_q \leq ApM_p/M_{p+1}$, $p \in \mathscr{N}$,

where $A$ and $B$ are constants independent on $p$.

We use two classes of ultradifferentiable functions, the Beurling class and Reumieu class (in short, $(M_p)$ class and $\{M_p\}$ class).

Let $u$ be a positive number and $\{u_p\}$ be a sequence of positive numbers

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increasing to $\infty$. We denote

$$H_p = \begin{Bmatrix} u^p, & \text{for the class } (M_p) \\ u_1 \cdots u_p, & \text{for the class } \{M_p\} \end{Bmatrix}. \tag{1}$$

Let $\Omega$ be an open set in $\mathscr{R}^n$.

By $\mathscr{D}_K^{H_p M_p}$ is denoted the Banach space of all $f \in \mathscr{E}^\infty(\mathscr{R}^n)$ with support in $K$ such that

$$\sup_{x \in K} |f^{(p)}(x)|/H_{|p|}M_{|p|} \to 0 \quad \text{as } p \to \infty$$

with the norm

$$q_{H_p M_p}(f) = \sup_{x \in K,\, p \in \mathscr{N}_0^n} |f^{(p)}(x)|/H_{|p|}M_{|p|}, \tag{2}$$

where $K$ is a compact set in $\mathscr{R}^n$. Then the basic spaces are

$$\mathscr{D}_K^{(M_p)} = \operatorname{proj}\lim_{u \to 0} \mathscr{D}_K^{u^p M_p}, \qquad \mathscr{D}_K^{\{M_p\}} = \operatorname{proj}\lim_{H_p} \mathscr{D}_K^{H_p M_p}$$

and

$$\mathscr{D}^*(\Omega) = \operatorname{ind}\lim_{K \Subset \Omega} \mathscr{D}_K^*,$$

where $*$ denotes either $(M_p)$ or $\{M_p\}$.

An operator of the form

$$P(D) = \sum_{|i| \geq 0} a_i D^i, \qquad a_i \in \mathscr{C},$$

is called the ultradifferential operator of class $(M_p)$ (of class $\{M_p\}$) if there are constants $L$ and $C$ (for every $L > 0$ there is a constant $C$) such that $|a_i| \leq CL^{|i|}/M_{|i|}$, $i \in \mathscr{N}_0^n$.

$A(\Omega)$ is the space of real analytic functions.

$B_r \equiv B(0, r)$ is the open ball with center at zero and with radius $r$.

**Theorem A** (see [4, Theorem 2.11]). *Let the sequence $M_p$ satisfy conditions* (M.1), (M.2) *and* (M.3). *For a given $H_p$ from* (1) *and a compact neighbourhood $Q$ of zero in $\mathscr{R}^n$ there exist an ultradifferential operator $P(D)$ of class $*$ and two functions $\varphi \in \mathscr{E}^\infty$ and $w \in D_Q^*$ such that*

$$P(D)\varphi = \delta + w, \qquad \operatorname{supp}\varphi \subset Q$$

*and*

$$\sup_{x \in Q} |\varphi^{(i)}(x)|/H_{|i|}M_{|i|} \to 0, \qquad |i| \to \infty.$$

## 3. Main result

**Theorem 1.** *Suppose that $M_p$ satisfies* (M.1), (M.2) *and* (M.3) *and that $\Omega$, $\Omega_1$ are two open sets in $\mathscr{R}^n$ such that $\Omega = \Omega_1 - B_r$ for some $r > 0$. An ultradistribution $T \in \mathscr{D}'^*(\Omega)$ is defined by the real analytic function $f$, $f \in \mathscr{A}(\Omega)$, if and only if $T \star w \in \mathscr{A}(\Omega_1)$ for every $w \in \mathscr{D}_K^*$, where $K = \overline{B}_r$ and $\star$ is the sign of convolution.*

*Proof.* If $T = f \in \mathscr{A}(\Omega)$, then it can be characterized as an infinitely differentiable function such that for every compact set $K' \subset \Omega$ and $p \in \mathscr{N}_0^n$ there exist two constants $C_{K'}$ and $C$ for which

$$\sup_{x \in K'} |f^{(p)}(x)| \leq Cp!C_{K'}^{|p|}, \qquad p \in \mathscr{N}_0^n.$$

Take any compact set $K'' \subset \Omega_1$. Then

$$\sup_{x\in K''} |(f \star w)^{(p)}(x)| \le Cp!C_{K'}^{|p|} \int_{\mathscr{R}^n} |w(x)|\, dx, \qquad p \in \mathscr{N}_0^n,$$

which proves that $f \star w \in \mathscr{A}(\Omega_1)$.

Suppose now that for $T \in \mathscr{D}'^*(\Omega)$ and for every $w \in \mathscr{D}_K^*$, $T \star w \in \mathscr{A}(\Omega_1)$ and $A$ is an open and relatively compact set, $\overline{A} \subset \Omega \subset \Omega_1$. In the first step of the proof we shall analyse the functional $R$,

$$R\colon (w, q) \to \sup_{x\in\overline{A}} \sqrt[|q|]{|(f \star w)^{(q)}(x)|/q!}, \qquad w \in \mathscr{D}_K^*,\, q \in \mathscr{N}^n,$$

which is related to the convergence radii of the power series

$$\sum_{|q|\ge 0} (f \star w)^{(q)}(x)(x-y)^q/q!, \qquad x \in A.$$

For a fixed $q \in \mathscr{N}^n$, $R$ is continuous in $w \in \mathscr{D}_K^*$, and for a fixed $w \in \mathscr{D}_K^*$, $R$ is bounded in $q \in \mathscr{N}^n$. Since $\mathscr{D}_K^*$ is barreled space, by the Banach theorem it follows that the family $\{R(\cdot, q), q \in \mathscr{N}^n\}$ is equicontinuous. Then for a fixed $L > 0$ there exist $\beta > 0$ and $H_p$ such that $R(w, q) < L$ when $w \in V_\beta$ and $q \in \mathscr{N}^n$, where $V_\beta$ is the neighbourhood of zero in $\mathscr{D}_K^*$ of the form

$$V_\beta = \left\{ \phi \in \mathscr{D}_K^*\colon \sup_{x\in K,\, p\in\mathscr{N}_0^n} |\phi^{(p)}(x)|/H_{|p|}M_{|p|} < \beta \right\}. \tag{3}$$

From the properties of the functional $R$ it follows that the family of functions

$$F_q\colon w \to (T \star w)^{(q)}/q!L^{|q|} = (D^q T \star w)/q!L^{|q|}, \qquad q \in \mathscr{N}_0^n,$$

is equicontinuous; $F_q$ maps $\mathscr{D}_K^*$ into $(\mathscr{CB})_A$. $(\mathscr{CB})_A$ is the space of continuous and bounded functions on $A$. Also, for every $w \in V_\beta$ and for every $q \in \mathscr{N}_0^n$, $(T * w)^{(q)}/q!L^{|q|}$ belongs to the ball $B(0, L) \subset (\mathscr{CB})_A$. Namely, there exists $C > 0$ such that

$$|(T \star w)^{(q)}(x)| \le Cq!L^{|q|}, \qquad x \in \overline{A},\, w \in V_\beta.$$

Denote by $\widetilde{\mathscr{D}}_K^{H_pM_p}$ the completion of $\mathscr{D}_K^*$ under the norm $q_{H_pM_p}$ given by (2), where $H_p$ is fixed by $V_\beta$. In the second part of the proof we shall show that the family $\{F_q\colon q \in \mathscr{N}_0^n\}$ can be extended by $\mathscr{D}_K^*$ to $\widetilde{\mathscr{D}}_K^{H_pM_p}$, keeping uniform continuity; let us denote this extension by $\{\overline{F}_q\colon q \in \mathscr{N}_0^n\}$.

For this purpose we use the theorem on the extension of a function by continuity (see [1, Chapter I, §8.5]). Let $\tilde{w} \in \widetilde{D}_K^{H_pM_p}$ and let $\{w_j\} \subset \mathscr{D}_K^*$ be the sequence which converges to $\tilde{w}$ in $\widetilde{\mathscr{D}}_K^{H_pM_p}$. We shall prove that $F_q(w_j)$ converges in $(\mathscr{CB})_A$ when $j \to \infty$, uniformly in $q \in \mathscr{N}_0^n$. To do this we shall show that $\{F_q(w_j), j \in \mathscr{N}\}$ is a Cauchy sequence in $(\mathscr{CB})_A$, uniform in $q \in \mathscr{N}_0^n$.

Let $W$ be a neighbourhood of zero in $(\mathscr{CB})_A$. Then it contains the ball $B(0, L) \subset (\mathscr{CB})_A$ for some $L > 0$. Consequently, $F_q(V_\beta) \subset W$, $q \in \mathscr{N}_0^n$, where $V_\beta$ is given by (3). Let $j_0$ be such that $w_i - w_j \in V_\beta$, $i, j \ge j_0$. Then

$$F_q(w_i) - F_q(w_j) = F_q(w_i - w_j) \in W, \qquad i, j \ge j_0,\; q \in \mathscr{N}_0^n,$$

and $F_q(w_j)$ converges to $h_q$ in $(\mathscr{CB})_A$, when $j \to \infty$, uniformly in $q \in \mathscr{N}_0^n$. By the cited extension theorem it follows that $\overline{F}_q(\tilde{w}) = h_q \in (\mathscr{CB})_A$, $q \in \mathscr{N}_0^n$; every $\overline{F}_q$, $q \in \mathscr{N}_0^n$, is uniquely defined.

We shall prove that

$$h_q = D^q(T \star \tilde{w})/q!L^{|q|} = (D^q T \star \tilde{w})/q!L^{|q|}, \qquad q \in \mathscr{N}_0^n.$$

The sequence $\{w_j\}$ converges to $\tilde{w}$ in $\mathscr{E}'^*_{B_r}$, as well. Therefore $(T \star w_j)^{(q)} = D^q T \star w_j$ converges to $D^q T \star \tilde{w}$ in $\mathscr{D}'^*_{\Omega_1}$, when $j \to \infty$, $q \in \mathscr{N}_0^n$ (see [6, Theorem 6.12]). Thus $(D^q T \star \tilde{w})/q!L^{|q|}$ must be $h_q$, $q \in \mathscr{N}_0^n$. Since the derivative is a continuous mapping of $\mathscr{D}'^*_{\Omega_1}$ into $\mathscr{D}'^*_{\Omega_1}$, it follows that $h_q = D^q(T \star \tilde{w})/q!L^{|q|}$, as well. Consequently, $T * \tilde{w} \in \mathscr{A}(\Omega_1)$.

In the third step of the proof it remains only to use Theorem A. Let us remark that if $\varphi \in C^\infty$ has support in the compact set $Q \Subset K$ and $q_{H_p M_p}(\varphi) < \infty$, then $\varphi \in \widetilde{\mathscr{D}}_K^{H_p M_p}$. We also know that $\mathscr{A}(\omega) \subset \mathscr{E}^*(\omega)$ and that $P(D)\mathscr{E}^*(\omega) \subset \mathscr{E}^*(\omega)$ for any open set $\omega \subset \mathscr{R}^n$. According to the above remark we deduce from Theorem A that

$$\frac{D^q T}{q!L^{|q|}} = P(D)\frac{(\varphi \star D^q T)}{q!L^{|q|}} + w \star D^q T \quad \text{on } \Omega, \;\; q \in \mathscr{N}_0^n.$$

Therefore $T \in \mathscr{A}(\Omega)$.

## 4. Applications of Theorem 1

Direct consequences of Theorem 1 are:

1. Lemma 2.4 in [4] we know the analytic form of the operator $P(D)$ and of the functions $\varphi$ and $w$ given in Theorem A. With these $P(D)$, $\varphi$ and $w$, Theorem 1 asserts that the equation $(P(D)X)(x) + (w \star f)(x) = f(x)$, $x \in \Omega$, has a solution $X = (\varphi * f) \in \mathscr{A}(\Omega)$ for any $f \in \mathscr{A}(\Omega)$.

2. Denote by $\delta_h$ the distribution $\delta$ shifted in the point $h$. The function $H: h \to \delta_h \star T$ maps $\mathscr{R}^n$ into $\mathscr{D}'^*(\mathscr{R}^n)$ and has all derivatives. Theorem 1 asserts that $H$ is real analytic if and only if $T$ is defined by a real analytic function. The property that $H$ is real analytic means that the set $\{D^q(\delta_h \star T)/q!L^{|q|}, \; q \in \mathscr{N}_0^n, h \in K\}$ is bounded in $D'^*(\mathscr{R}^n)$ for every compact set $K \subset \mathscr{R}^n$ and an $L > 0$ which depends on $K$, namely that

$$\sup_{h \in K, q \in \mathscr{N}_0^n} |D^q(T \star w)(h)|/q!L^{|q|} < C \quad \text{for every } w \in \mathscr{D}^*(\mathscr{R}^n)$$

where $C > 0$ depends on $K$ and $w \in \mathscr{D}^*(\mathscr{R}^n)$.

*Application of Theorem* 1 *to convolution equations.* Let

$$A \star T \equiv \sum_{k=1}^{m} A_{j,k} \star T_k, \qquad j = 1, \ldots, m, \tag{4}$$

where $A = (A_{j,k})$ is a given $(m \times m)$-matrix of elements belonging to $\mathscr{E}'^*(\mathscr{R}^n)$ and $T$ is an $m$-tuple $\{T_1, \ldots, T_m\}$ of unknown ultradistributions.

**Theorem 2.** *Suppose that the system $T \star A = 0$ has the following property: Any solution which belongs to $(\mathscr{E}^*(\Omega_1))^m$ belongs to $(\mathscr{A}(\Omega_1))^m$ as well. Then for* *every m-tuple U of ultradistributions which is a solution of the system* $T \star A = 0$ *there exists* $r > 0$ *such that* $U \in (\mathscr{A}(\Omega))^m$, $\Omega = \Omega_1 - B_r$.

*Proof.* For the sake of simplicity we shall prove Theorem 2 only in case $m = 1$. Suppose that $U \in \mathscr{D}'^*(\Omega)$ is a solution to equation $T \star A = 0$. Then $U \star w \in \mathscr{E}^*(\Omega_1)$ for every $w \in \mathscr{D}^*(\overline{B}_r)$ satisfies equation $T \star A = 0$ as well, because of properties of the convolution. Since $U \star w \in \mathscr{E}^*(\Omega_1)$, by the supposition in Theorem 2, $U \star w$ belongs to $\mathscr{A}(\Omega_1)$. Theorem 1 asserts that $U \in \mathscr{A}(\Omega)$. This is the end of the proof.

*Application of Theorem* 1 *to Painlevé's theorem.* Let $V$ be an open set in $\mathscr{C}$ and $\Omega = V \cap \mathscr{R}$. Classical Painlevé's theorem asserts that a function $f$ holomorphic on $V \backslash \Omega$ and continuous on $V$ is holomorphic on the whole set $V$. It is well known that the continuity can be replaced by the existence and equality of the limits $\lim_{y \to 0^+} f(x \pm iy)$ in $\mathscr{D}'(\Omega)$. Theorem 1 admits to weaken this condition, supposing that the above limits exist in $\mathscr{D}'^*(\Omega)$.

**Theorem 3.** *Let* $f$ *be a holomorphic function on* $V \backslash \Omega$. *If the limits* $f(x \pm i0) = \lim_{y \to 0^+} f(x \pm iy)$ *exist in* $\mathscr{D}'^*(\Omega)$ *and* $f(x + i0) = f(x - i0)$, *then* $f$ *is holomorphic on* $V$.

*Proof.* The method of the proof is just the same as for distributions. First we have to apply classical Painlevé's theorem to the convolution $f \star w$, $w \in \mathscr{D}_K^*$, and then use Theorem 1.

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Institute of Mathematics, University of Novi Sad, 21000 Novi Sad, Yugoslavia