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ON AN ELLIPTIC EQUATION WITH CONCAVE AND CONVEX NONLINEARITIES

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ABSTRACT. We study the semilinear elliptic equation $-\Delta u = \lambda|u|^{q-2}u + \mu|u|^{p-2}u$ in an open bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions; here $1 < q < 2 < p < 2^*$. Using variational methods we show that for $\lambda > 0$ and $\mu \in \mathbb{R}$ arbitrary there exists a sequence (v_k) of solutions with negative energy converging to 0 as $k \to \infty$. Moreover, for $\mu > 0$ and λ arbitrary there exists a sequence of solutions with unbounded energy. This answers a question of Ambrosetti, Brézis and Cerami. The main ingredient is a new critical point theorem, which guarantees the existence of infinitely many critical values of an even functional in a bounded range. We can also treat strongly indefinite functionals and obtain similar results for first-order Hamiltonian systems.

1. INTRODUCTION

We consider the semilinear elliptic problem

$$(1) \qquad \begin{aligned} -\Delta u &= \lambda|u|^{q-2}u + \mu|u|^{p-2}u \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary and $1 < q < 2 < p < 2^* := 2N/(N-2)$; we set $2^* = \infty$ if $N = 1, 2$. In [1] Ambrosetti et al. showed that for $\lambda > 0$ small and $\mu > 0$ there exist infinitely many solutions $u \in H_0^1(\Omega)$ of (1) with negative energy

$$\phi_{\lambda,\mu}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{\mu}{p}\int_\Omega |u|^p\,dx$$

and infinitely many solutions with positive energy. The goal of this note is to show that the restriction on λ is not needed. In fact, we shall prove the following theorem which gives a positive answer to problem (c) of [1].

Theorem 1. *Assume that* $1 < q < 2 < p < 2^*$.

(a) *For every* $\mu > 0$, $\lambda \in \mathbb{R}$, *problem* (1) *has a sequence of solutions* (u_k) *such that* $\phi_{\lambda,\mu}(u_k) \to \infty$ *as* $k \to \infty$.

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(b) *For every* $\lambda > 0$, $\mu \in \mathbb{R}$, *problem* (1) *has a sequence of solutions* (v_k) *such that* $\phi_{\lambda,\mu}(v_k) < 0$ *and* $\phi_{\lambda,\mu}(v_k) \to 0$ *as* $k \to \infty$.

We do not know whether $v_k \to 0$ as $k \to \infty$. This is the case if 0 is the only solution of (1) with energy 0. However, the following holds.

Proposition 1. (a) *For* $\lambda \in \mathbb{R}$ *and* $\mu \leq 0$ *there are no solutions with positive energy. Moreover*

$$\inf\{\|u\| : u \text{ solves } (1), \phi_{\lambda,\mu}(u) > 0\} \to \infty \quad \text{as } \mu \to 0^+.$$

(b) *For* $\mu \in \mathbb{R}$ *and* $\lambda \leq 0$ *there are no solutions with negative energy. Moreover*

$$\sup\{\|u\| : u \text{ solves } (1), \phi_{\lambda,\mu}(u) \leq 0\} \to 0 \quad \text{as } \lambda \to 0^+.$$

Theorem 1 remains true if the special nonlinearity in (1) is replaced by an odd function $f(x, u)$ which behaves near 0 asymptotically (and uniformly in x) like $\lambda|u|^{q-2}u$, $\lambda \neq 0$, and which is superquadratic and subcritical near infinity. In [1] and [7] the existence of solutions with negative energy has also been proved in the critical case $p = 2^*$ provided $\lambda > 0$ is small enough. It is not clear whether this remains true for large λ because the local Palais-Smale condition $(\mathrm{PS})_c$ fails for large λ even if $c < 0$; see [7] for a discussion of this.

Whereas the first statement of Theorem 1 can be proved using well-known critical point theorems for even functionals (cf. [2, 3, 8]), we shall need a new critical point theorem in order to obtain the sequence (v_k). Since it does not require more effort, we shall treat more general symmetries than the $\mathbb{Z}/2$-symmetry in our abstract result and we allow the functional to be strongly indefinite. This allows us to apply our critical point theorem also to first-order Hamiltonian systems having a similar type of nonlinearity.

2. The abstract critical point theorem

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. We are interested in multiple critical points of φ. For this we need a symmetry condition on φ which contains even functionals as a special case. In order to formulate this we have to recall a certain class of admissible representations. Let G be a compact Lie group and V a finite-dimensional orthogonal representation of G. Then V is said to be admissible if the following Borsuk-Ulam type condition holds.

(*) Every continuous equivariant map $h : \overline{\mathscr{O}} \to V^k$ where $\mathscr{O}$ is an open, bounded and invariant neighbourhood of 0 in V^{k+1}, $k \geq 1$, has a zero in $\partial\mathscr{O}$.

Here $\mathscr{O}$ is invariant if $gv := (gv_1, \dots, gv_{k+1}) \in \mathscr{O}$ for every $g \in G$ and $v = (v_1, \dots, v_{k+1}) \in \mathscr{O}$. The map h is equivariant if $h(gv) = gh(v)$. The classical theorem of Borsuk says that $V = \mathbb{R}$ with the antipodal representation of $G = \mathbb{Z}/2$ is admissible. Admissible representations can be classified completely using an algebraic criterion (see [4, Theorem 3.7]). From the results of [4] it also follows that admissible representations are precisely those which have the "dimension property" used by Benci [6].

It is clear that an admissible representation V cannot have nontrivial fixed points, that is,

$$V^G := \{v \in V : gv = v \text{ for all } g \in G\} = \{0\}.$$

We can now state our conditions on ϕ.

(A_1) There exists an admissible representation V of G such that $X = \bigoplus_{j\in I} X(j)$ with $I = \mathbb{N}$ or $I = \mathbb{Z}$ and $X(j) \cong V$ for every $j \in I$. The space X is then a Banach space with isometric linear G-action. The functional $\phi: X \to \mathbb{R}$ is invariant under this action: $\phi(gu) = \phi(u)$ for $g \in G$ and $u \in X$.

(A_2) For every $k \geq k_0$ there exists $R_k > 0$ such that $\phi(u) \geq 0$ for every $u \in X_k := \bigoplus_{j\geq k} X(j)$ with $\|u\| = R_k$.

(A_3) $b_k := \inf_{u\in B_k} \varphi(u) \to 0$ as $k \to \infty$. Here $B_k := \{u \in X_k : \|u\| \leq R_k\}$.

(A_4) For every $k \geq 1$ there exists $r_k \in (0, R_k)$ and $d_k < 0$ such that $\phi(u) \leq d_k$ for every $u \in X^k := \bigoplus_{j\leq k} X(j)$ with $\|u\| = r_k$.

(A_5) Every sequence $u_n \in X^n_{-n} := \bigoplus_{j=-n}^{n} X(j)$ with $\phi(u_n) < 0$ bounded and $(\phi|X^n_{-n})'(u_n) \to 0$ as $n \to \infty$ has a subsequence which converges to a critical point of ϕ.

Observe that (A_3) and (A_4) imply $b_k \leq d_k < 0$.

Theorem 2. *If $\phi \in C^1(X, \mathbb{R})$ satisfies (A_1)–(A_5), then for each $k \geq k_0$, φ has a critical value $c_k \in [b_k, d_k]$, hence $c_k \to 0$ as $k \to \infty$.*

Proof. We fix $n \geq k \geq k_0$ and define an almost critical value c_k^n as follows. Setting

$$X_k^n := \bigoplus_{j=k}^{n} X(j)$$

and

$$B_k^n := \{u \in X_k^n : \|u\| \leq R_k\}$$

we define

$$\Gamma_k^n := \{\gamma \in \mathscr{C}(B_k^n, X^n_{-n}) : \gamma \text{ is equivariant}, \ \gamma(u) = u \text{ if } \|u\| = R_k\}$$

and

$$c_k^n := \sup_{\gamma\in\Gamma_k^n} \min_{u\in B_k^n} \phi(\gamma(u)).$$

We shall show that $b_k \leq c_k^n \leq d_k < 0$ for every $n \geq k$. By the quantitative deformation lemma (see [10]) c_k^n is an almost critical value of $\phi|X^n_{-n}$, that is, there exists a sequence $u_i \in X^n_{-n}$ with $\phi(u_i) \to c_k^n$ and $(\phi|X^n_{-n})'(u_i) \to 0$ as $i \to \infty$. Using (A_5) we see that c_k^n converges along a subsequence to a critical value $c_k \in [b_k, d_k]$ as $n \to \infty$.

It remains to prove $c_k^n \in [b_k, d_k]$ for every $n \geq k$. The inequality $c_k^n \geq b_k$ is obvious from the definitions. In order to see $c_k^n \leq d_k$ it suffices to show that for every $\gamma \in \Gamma_k^n$ there exists $u \in B_k^n$ satisfying $\gamma(u) \in X^k$ and $\|\gamma(u)\| = r_k$. Given $\gamma \in \Gamma_k^n$ we set

$$\mathscr{O} := \{u \in B_k^n : \|\gamma(u)\| < r_k\}.$$

The equivariance of γ implies $\gamma(0) = 0$ because 0 is the only fixed point of the action. Therefore $\mathscr{O}$ is an open invariant neighbourhood of 0 and $\overline{\mathscr{O}} \subset \operatorname{int} B_k^n$. Let $P: X^n_{-n} \to X^n_{k+1}$ be the projection along X^k_{-n} and set $h := P\circ\gamma : \overline{\mathscr{O}} \to X^n_{k+1}$. Since $X_k^n \cong V^{n-k+1}$ and $X^n_{k+1} \cong V^{n-k}$, the admissibility of V gives us a point $u \in \partial\mathscr{O}$ with $h(u) = 0$. Clearly this implies $\|\gamma(u)\| = r_k$ and $\gamma(u) \in X^k_{-n} \subset X^k$ as required. □

Remark. Theorem 2 can be considered as a dual version of Theorem 2.5 of [3] or Theorem 3.1 of [5].

3. Proofs

Proof of Theorem 1. We first prove the existence of (v_k) and assume $\lambda > 0$. We set $X := H_0^1(\Omega)$ with

$$\|u\| := \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2}.$$

Let (e_j) be any orthonormal base of X and set $X(j) := \operatorname{span}(e_j)$. The functional

$$\begin{aligned}\phi(u) &:= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{\mu}{p}\int_\Omega |u|^p \, dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\|u\|_q^q - \frac{\mu}{p}\|u\|_p^p\end{aligned}$$

is well defined on X for $1 < q < 2 < p < 2^*$. Since ϕ is even, assumption (A_1) is obviously satisfied with $G = \mathbb{Z}/2$ acting on $V = \mathbb{R}$ via the antipodal map.

In order to see (A_2) we set

$$\mu_k := \sup_{u \in X_k - 0} \|u\|_q / \|u\|.$$

It follows easily from the Rellich embedding theorem that $\mu_k \to 0$ as $k \to \infty$. Choose $c_1 > 0$ such that $\|u\|_p^p \le c_1 \|u\|^p$. We obtain for $u \in X_k$

$$\phi(u) \ge \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\mu_k^q \|u\|^q - |\mu| \frac{c_1}{p}\|u\|^p.$$

Since $p > 2$, we have

$$|\mu| \frac{c_1}{p}\|u\|^p \le \frac{1}{4}\|u\|^2$$

for $\|u\| \le R$, $R > 0$ small. Now we set $R_k := (4\lambda\mu_k^q/q)^{1/(2-q)}$ so that

$$\frac{1}{4}R_k^2 = \frac{\lambda}{q}\mu_k^q R_k^q.$$

Clearly $R_k \to 0$, so there exists k_0 with $R_k \le R$ when $k \ge k_0$. Thus if $u \in X_k$, $k \ge k_0$, satisfies $\|u\| = R_k$ we have

$$\phi(u) \ge \frac{1}{4}\|u\|^2 - \frac{\lambda}{q}\mu_k^q \|u\|^q = 0.$$

This proves (A_2). Next (A_3) follows immediately from $R_k \to 0$. Also (A_4) is evident because X^k is finite dimensional, hence all norms on X^k are equivalent. Therefore the term $\frac{-\lambda}{q}\|u\|_q^q$ dominates near 0. This is precisely the point where $\lambda > 0$ enters. Finally, the Palais-Smale condition (A_5) can be shown as in [8] or [10].

The existence of the sequence (u_k) follows from the symmetric mountain pass theorem of [2, 8] or the fountain theorem of [3, 10]. □

Proof of Proposition 1. (a) Fix $\lambda, \mu \in \mathbb{R}$. From $\phi'_{\lambda,\mu}(u) = 0$ and $\phi_{\lambda,\mu}(u) \geq 0$ one obtains easily

$$\left(\frac{1}{2} - \frac{1}{q}\right) \|u\|^2 + \left(\frac{1}{q} - \frac{1}{p}\right) \mu \|u\|_p^p \geq 0.$$

Since $1 < q < 2 < p$, we see immediately that for $\mu \leq 0$ only $u = 0$ is a solution with nonnegative energy. If $\mu > 0$, then there are positive constants c_1, c_2 with

$$-c_1 \|u\|^2 + \mu c_2 \|u\|^p \geq 0,$$

hence

$$\|u\|^{p-2} \geq \mu^{-1} c_1/c_2 \to \infty \quad \text{as } \mu \to 0^+.$$

(b) Similarly, from $\phi'_{\lambda,\mu}(v) = 0$ and $\phi_{\lambda,\mu}(v) \leq 0$ one obtains

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^2 + \left(\frac{1}{p} - \frac{1}{q}\right) \lambda \|u\|_q^q \leq 0.$$

This implies that for $\lambda \leq 0$ only $u = 0$ is a solution with nonpositive energy. For $\lambda > 0$ there are positive constants c_3, c_4 with

$$c_3 \|u\|^2 - \lambda c_4 \|u\|^q \leq 0,$$

hence

$$\|u\|^{2-q} \leq \lambda c_4/c_3 \to 0 \quad \text{as } \lambda \to 0^+. \quad \square$$

4. Hamiltonian systems

We consider the Hamiltonian system with periodic conditions

$$\begin{aligned} \dot{u}(t) &= J \nabla H_{\lambda,\mu}(u(t)), \\ u(0) &= u(1), \end{aligned} \tag{2}$$

where $H_{\lambda,\mu}$ is defined on $\mathbb{R}^2$ by

$$H_{\lambda,\mu}(u) := \frac{\lambda}{q} |u|^q + \frac{\mu}{p} |u|^p$$

and $1 < q < 2 < p < \infty$. We denote by J the symplectic matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Following the Poincaré principle, the solutions of (2) are the critical points of the functional

$$\phi_{\lambda,\mu}(u) := \frac{1}{2} \langle \dot{u}, Ju \rangle - \int_{\mathbb{T}} H_{\lambda,\mu}(u(t))\, dt$$

defined on the Sobolev space $X := H^{1/2}(\mathbb{T}, \mathbb{R}^2)$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The bracket is the duality bracket between $H^{-1/2}(\mathbb{T}, \mathbb{R}^2)$ and X.

Theorem 3. *Assume that* $1 < q < 2 < p < \infty$.

(a) *For every* $\mu > 0$, $\lambda \in \mathbb{R}$, *problem* (2) *has a sequence of solutions* (u_k) *such that* $\phi_{\lambda,\mu}(u_k) \to \infty$ *as* $k \to \infty$.

(b) *For every* $\lambda > 0$, $\mu \in \mathbb{R}$, *problem* (2) *has a sequence of solutions* (v_k) *such that* $\phi_{\lambda,\mu}(v_k) < 0$ *and* $\phi_{\lambda,\mu}(v_k) \to 0$ *as* $k \to \infty$.

Proof. Part (a) is a particular case of Theorem 4.1 in [5]. We prove part (b) by using Theorem 2. We assume $\lambda > 0$ and set $X(j) := \operatorname{span}(e^{2\pi j t J} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$. Each function $u \in X$ has a Fourier expansion

$$u \sim \sum_{j \in \mathbb{Z}} e^{2\pi j t J} \hat{u}(j)$$

and

$$\sum_{j\in\mathbb{Z}} |j| \cdot |\hat{u}(j)|^2 < \infty.$$

On X we choose the inner product

$$(u, v) := 2\pi \sum_{j\in\mathbb{Z}} |j| \cdot \hat{u}(j)\overline{\hat{v}(j)} + \hat{u}(0)\overline{\hat{v}(0)}$$

and the corresponding norm $\|u\| := \sqrt{(u, u)}$. On each $X(j)$ we consider the antipodal action of $\mathbb{Z}/2$ so that (A_1) is satisfied.

It is easy to verify (A_2) and (A_3) as in the proof of Theorem 1. To prove (A_4), we use the orthogonal decomposition $X = X^+ \oplus X^0 \oplus X^-$ which refers to the subspaces with $j \geq 1$, $j = 0$, $j \leq -1$ respectively. On X^k, we have for $\|u\|$ small enough

$$\begin{aligned}\phi(u) &\leq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{\lambda}{q}\|u\|_q^q + \frac{|\mu|}{p}\|u\|_p^p \\ &\leq \frac{1}{2}c\|u^+\|_q^2 - \frac{1}{2}\|u^-\|^2 - \frac{\lambda}{2q}\|u\|_q^q.\end{aligned}$$

We have used the fact that $X^k \cap X^+$ is finite dimensional.

Since $\|u^+\|_q \leq c_1\|u\|_q$ on $X^k \cap X^+$ and since $q < 2$, we obtain finally for $\|u\|$ small enough,

$$\begin{aligned}\phi(u) &\leq c_2\|u\|_q^2 - \frac{1}{2}\|u^-\|^2 - \frac{\lambda}{2q}\|u\|_q^q \\ &\leq -\frac{1}{2}\|u^-\|^2 - c_3\|u\|_q^q\end{aligned}$$

where $c_3 > 0$. We choose $r_k \in (0, R_k)$ small enough so that the above inequality applies for $\|u\| = r_k$. Then, if $\|u\| = r_k$ and $\|u^-\| \geq r_k/4$, we have

$$\phi(u) \leq -r_k^2/32.$$

If $\|u\| = r_k$ and $\|u^-\| \leq r_k/4$, we have

$$\phi(u) \leq -c_3(\|u^+ + u^0\|_q - \|u^-\|_q)^q \leq -c_3(r_k/2)^q.$$

Hence assumption (A_4) is satisfied.

If $\mu \neq 0$, the proof of the Palais-Smale condition can be found in [10]. If $\mu = 0$, the argument is similar. □

Remark. Theorem 3 remains true if we consider the Hamiltonian

$$H_{\lambda,\mu}(u) = a(t)\left(\frac{\lambda}{q}\mu|u|^q + \frac{\mu}{p}|u|^p\right)$$

where $a \in \mathscr{C}(\mathbb{T}, \mathbb{R})$ is positive.

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