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UNIQUENESS OF POSITIVE SOLUTIONS OF NONLINEAR SECOND-ORDER EQUATIONS

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ABSTRACT. In this paper we study the uniqueness question of positive solutions of the two-point boundary value problem: $u''(t) + f(|t|, u(t)) = 0$, $-R < t < R$, $u(\pm R) = 0$ where $R > 0$ is fixed and $f\colon [0, R] \times [0, \infty) \to \mathbb{R}$ is in $C^1([0, R] \times [0, \infty))$. A uniqueness result is proved when f satisfies some appropriate conditions. Some examples illustrating our theorem are also given.

1. INTRODUCTION

In this paper we study the question of uniqueness for solutions $u \in C^2([-R, R])$ of the two-point boundary value problem

$$(1.1)\qquad \begin{cases} u''(t) + f(|t|, u(t)) = 0, & -R < t < R, \\ u(\pm R) = 0, & \\ u(t) > 0, & -R < t < R, \end{cases}$$

where $R > 0$ is fixed and the function $f\colon [0, R]\times[0, \infty) \to \mathbb{R}$ is in $C^1([0, R]\times [0, \infty))$ and satisfies the following hypotheses:

(H_1) $uf_u(t, u) > f(t, u)$ for $(t, u) \in [0, R] \times (0, \infty)$.
(H_2) $f_t(t, u) \le 0$ for $(t, u) \in [0, R] \times [0, \infty)$.
(H_3) There exists $u > 0$ such that $f(R, u) \ge 0$.

Conditions for the existence of solutions of (1.1) were studied by many authors: see for instance De Figueiredo, Lions and Nussbaum [2], Granas, Guenther and Lee [4], Lions [8], Rabinowitz [10] and the references therein.

The uniqueness question for problem (1.1) was studied in the case where $f(|t|, u)$ is nonnegative, nondecreasing in u and concave in u in the following generalized sense: for $u > 0$ and arbitrary $\tau \in (0, 1)$, $f(|t|, \tau u) - \tau f(|t|, u) > 0$ for $t \in (-R, R)$ (see for instance Krasnoselskii [6]). A uniqueness result was also established by Ni and Nussbaum [9] for a different nonlinearity. When $f(|t|, u)$ is nonincreasing in u, uniqueness is easily obtained: see Granas, Guenther and Lee [4]. Recently, in the particular case where $f(|t|, u) = p(|t|)f(u)$, several uniqueness results have been established in [1] under weaker regularity assumptions.

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We shall prove the following theorem.

Theorem 1. *Let* $f \in C^1([0, R] \times [0, \infty))$ *satisfy* $(\mathrm{H}_1) - (\mathrm{H}_3)$. *Then problem* (1.1) *has at most one solution in* $C^2([-R, R])$.

Remark 1. It follows from a result of Gidas, Ni and Nirenberg [3] that under the assumption (H_2) any solution $u \in C^2([-R, R])$ of problem (1.1) is necessarily symmetric about the origin and such that $u' < 0$ on $(0, R)$. Therefore solutions of (1.1) can be treated as positive solutions of

$$\text{(1.2)} \qquad \begin{cases} u''(t) + f(t, u(t)) = 0, & 0 \le t < R, \\ u(R) = u'(0) = 0. \end{cases}$$

The key ingredient is the use of the function

$$\varphi(t, \alpha) = \frac{\partial u}{\partial \alpha}(t, \alpha)$$

where u is considered as a function of both t and the parameter $\alpha = u(0)$. The main point is to show that φ changes sign only once in $[0, R]$ and that $\varphi(R, u(0)) < 0$.

The function φ has been used in cases where f in (1.2) also depends on u' and even when f is singular at $t = 0$. For instance Kwong [7] established uniqueness for the following boundary value problems:

$$u''(t) + \frac{m}{t}u'(t) + u(t)^p - u(t) = 0, \qquad u(t) > 0, 0 \le a < t < b \le \infty,$$
$$u'(a) = 0,$$
$$u(b) = 0 \quad \text{if } b < \infty$$

or

$$\lim_{t \to \infty} u(t) = 0 \quad \text{if } b = \infty$$

with $m \ge 0$, $p > 1$ and $p < (m+3)/(m-1)$ if $a = 0$ and $m > 1$.

In Section 2 we introduce an initial value problem for the differential equation in (1.2) and we state our main result—Theorem 2—from which we derive Theorem 1. Section 3 is devoted to the proof of Theorem 2: some crucial lemmas are needed. Finally some examples to which our theorem applies are given in Section 4.

2. Main results

Let $f \in C^1([0, R] \times [0, \infty))$. If we set

$$f(t, u) = f(R, u) \quad \text{for } t > R$$

and

$$f(t, u) = f(t, 0) + u f_u(t, 0) \quad \text{for } u < 0,$$

then we can assume that $f \in C([0, \infty) \times (-\infty, \infty))$ and that f is continuously differentiable in u. Moreover if (H_2) is verified we can assume that f is nonincreasing in $t \ge 0$ when $u \ge 0$.

Now we introduce the initial value problem

$$\text{(2.1)} \qquad u''(t) + f(t, u(t)) = 0, \qquad t > 0,$$

$$\text{(2.2)} \qquad u(0) = \alpha, \qquad u'(0) = 0$$

where $\alpha > 0$ is a parameter. The unique solution of this problem will be denoted by $u(\cdot, \alpha)$ or u. $u(\cdot, \alpha)$ is defined on a maximum interval $[0, T_\alpha)$ ($T_\alpha = \infty$ possibly). We define $t(\alpha)$ by

$$t(\alpha) = \sup\{t \in [0, T_\alpha)\,;\, u(s, \alpha) > 0 \text{ for } s \in [0, t)\}.$$

If $u(t, \alpha) > 0$ for all $t \in [0, T_\alpha)$, then $t(\alpha) = T_\alpha$ and if $u(\cdot, \alpha)$ has a finite zero in $[0, T_\alpha)$, $t(\alpha)$ denotes the first zero of $u(\cdot, \alpha)$ in $[0, T_\alpha)$.

Remark 2. When $t(\alpha) < T_\alpha$ we have $u(t(\alpha), \alpha) = 0$. Under the assumption (H_2) the solution $u(\cdot, \alpha)$ of problem (2.1), (2.2) is such that $u'(t, \alpha) < 0$ for $t \in (0, t(\alpha))$: as before this follows from a result of Gidas, Ni and Nirenberg [3].

We introduce the following definition.

Definition. The solution $u(\cdot, \alpha)$ of problem (2.1), (2.2) is said to be admissible if

(i) $t(\alpha) \le R$;
(ii) $u'(t, \alpha) \le 0$ for $t \in [0, t(\alpha))$;
(iii) $f(0, \alpha) > 0$.

Remark 3. Let $u(\cdot, \alpha)$ be an admissible solution of problem (2.1), (2.2). (ii) implies that $u(\cdot, \alpha)$ is bounded on $[0, t(\alpha))$. Therefore $t(\alpha) < T_\alpha$ and hence $u(t(\alpha), \alpha) = 0$.

Now we can state our main result.

Theorem 2. *Let* $f \in C^1([0, R] \times [0, \infty))$ *satisfy* $(H_1) - (H_3)$. *If the solution* $u(\cdot, \alpha_0)$ *of problem* (2.1), (2.2) *is admissible for some* $\alpha_0 > 0$, *then we have*:

(i) $u(\cdot, \alpha)$ *is admissible for all* $\alpha \in [\alpha_0, \infty)$;
(ii) $t(\alpha)$ *is a strictly decreasing function of* $\alpha \in [\alpha_0, \infty)$.

The proof of Theorem 1 is now easy. Let $u \in C^2([-R, R])$ be a solution of problem (1.1). By Remark 1 u can be treated as a positive solution of problem (1.2). Therefore u is the solution of problem (2.1), (2.2) such that $\alpha = u(0)$ and $t(\alpha) = R < T_\alpha$. We first show that u is admissible. By Remark 1 again it is enough to show that $f(0, u(0)) > 0$. If not, (H_2) implies that $f(t, u(0)) \le 0$ for all $t \in [0, R]$. Then by (H_1) we deduce that $f(t, u) < 0$ for $t \in [0, R]$ and $u \in (0, u(0))$. Thus $u'' > 0$ on $(0, R)$ from which we get $u' > 0$ on $(0, R)$ and we reach a contradiction. Now let $v \in C^2([-R, R])$ be another solution of problem (1.1) and assume for instance that $u(0) < v(0)$. Since $t(u(0)) = t(v(0)) = R$, we get a contradiction with (ii) in Theorem 2.

3. Proof of Theorem 2

We first state and prove two lemmas (1 and 4) which are needed in the proof of Theorem 2.

Lemma 1. *Assume* (H_1) *and* (H_2). *Let* $u(\cdot, \alpha)$ *be an admissible solution of problem* (2.1), (2.2). *Then* $\varphi = \varphi(\cdot, \alpha) = \frac{\partial u}{\partial \alpha}(\cdot, \alpha)$ *is such that* $\varphi(t(\alpha), \alpha) < 0$.

We shall need two lemmas.

Lemma 2. *Assume* (H_1) *and* (H_2). *Let* $u(\cdot, \alpha)$ *be an admissible solution of problem* (2.1), (2.2). *Then*

(i) $t(\alpha) < T_\alpha$, $u(t(\alpha), \alpha) = 0$ *and* $u'(t, \alpha) < 0$ *for* $t \in (0, t(\alpha))$;
(ii) $(u'/u)' < 0$ *on* $(0, t(\alpha))$.

Proof. (i) is given by Remark 3 and Remark 2. We now prove (ii). We have

$$u(t)^2(u'/u)'(t) = u''(t)u(t) - u'(t)^2 = -(u(t)f(t, u(t)) + u'(t)^2) \tag{3.1}$$

for $t \in (0, t(\alpha))$. Denoting by $\psi(t)$ the right-hand side in (3.1) we get

$$\psi'(t) = -\{u(t)f_t(t, u(t)) + (u(t)f_u(t, u(t)) - f(t, u(t)))u'(t)\}$$

for $t \in (0, t(\alpha))$. Using (i), (H_1) and (H_2) we obtain $\psi'(t) > 0$ for $t \in (0, t(\alpha))$ and hence $\psi(t) < \psi(t(\alpha)) = -u'(t(\alpha), \alpha)^2 \le 0$ for $t \in (0, t(\alpha))$ and the lemma follows.

Lemma 3. *Assume* (H_1) *and* (H_2). *Let* $u(\cdot, \alpha)$ *be an admissible solution of problem* (2.1), (2.2). *Then* $\lim_{t\to t(\alpha)^-}(u'/u)(t) = -\infty$.

Proof. The lemma is clear when $u'(t(\alpha), \alpha) < 0$. Now assume that $u'(t(\alpha), \alpha) = 0$. Then necessarily $f(t(\alpha), 0) < 0$. Indeed (H_1) implies that $f(t, 0) \le 0$ for $t \in [0, t(\alpha)]$. Thus, if $f(t(\alpha), 0) = 0$, we deduce from (H_2) that $f(t, 0) = 0$ for $t \in [0, t(\alpha)]$. Therefore the uniqueness theorem for the initial value problem for ordinary differential equations implies that $u = 0$ on $[0, t(\alpha)]$ and we reach a contradiction. Now we have $u''(t(\alpha), \alpha) = -f(t(\alpha), 0) > 0$ and the lemma follows.

Proof of Lemma 1. For $\beta \ge 0$ we define the function $u_\beta = u + \beta u'$. By Lemmas 2 and 3, for any $\beta > 0$ there exists a unique $r(\beta) \in (0, t(\alpha))$ such that

$$u_\beta > 0 \quad \text{on } [0, r(\beta)) \qquad \text{and} \qquad u_\beta < 0 \quad \text{on } (r(\beta), t(\alpha)).$$

Moreover $r(\beta)$ is a strictly decreasing continuous function of β satisfying

$$r(0) = t(\alpha) \qquad \text{and} \qquad r(\beta) \to 0 \quad \text{as } \beta \to \infty. \tag{3.2}$$

Differentiating (2.1) with respect to α and using (2.2), we obtain

$$\varphi''(t) + f_u(t, u(t))\varphi(t) = 0 \quad \text{for } t \in [0, t(\alpha)), \tag{3.3}$$

$$\varphi(0) = 1, \qquad \varphi'(0) = 0. \tag{3.4}$$

We can write

$$\begin{aligned} u'(t(\alpha))\varphi(t(\alpha)) &= \int_0^{t(\alpha)} \{u''(t)\varphi(t) - \varphi''(t)u(t)\}\, dt \\ &= \int_0^{t(\alpha)} \varphi(t)\{u(t)f_u(t, u(t)) - f(t, u(t))\}\, dt. \end{aligned}$$

Assume that $\varphi(t) > 0$ for $t \in [0, t(\alpha))$. Then, by (H_1) the right-hand side is positive. Since the left-hand side is nonpositive, we obtain a contradiction. Thus we can define t_1 to be the first zero of φ in $(0, t(\alpha))$. We shall show that t_1 is the unique zero of φ in $(0, t(\alpha)]$. Suppose the contrary. Then we denote by t_2 the first zero of φ in $(t_1, t(\alpha)]$. Using (3.2) we can choose $\beta > 0$ such that $r(\beta) \leq t_1$. Then we write

$$\begin{aligned} &-\varphi'(t_2)u_\beta(t_2) + \varphi'(t_1)u_\beta(t_1) \\ &\quad = \int_{t_1}^{t_2} \{u''_\beta(t)\varphi(t) - \varphi''(t)u_\beta(t)\}\,dt \\ &\quad = \int_{t_1}^{t_2} \varphi(t)\{u(t)f_u(t, u(t)) - f(t, u(t)) - \beta f_t(t, u(t))\}\,dt. \end{aligned}$$

By (H_1), (H_2) and the fact that $\varphi < 0$ on (t_1, t_2) the right-hand side is negative. Since the left-hand side is positive, we obtain a contradiction. Thus t_1 is the unique zero of φ in $(0, t(\alpha)]$ and the lemma follows.

Lemma 4. *Assume* (H_1) *and* (H_2). *Let* $u(\cdot, \alpha)$ *be an admissible solution of problem* (2.1), (2.2). *Then there exists* $\eta > 0$ *such that* $u(\cdot, \gamma)$ *is admissible for all* $\gamma \in [\alpha, \alpha + \eta)$.

Proof. Let $\gamma > \alpha$ with $\gamma - \alpha$ sufficiently small. By Remark 3 we have $t(\alpha) < T_\alpha$. Since $\alpha \to T_\alpha$ is a lower semicontinuous function (see Hartman [5] Theorem 2.1 p. 94), $t(\alpha) < T_\gamma$. Now we have

$$u(t(\alpha), \gamma) = (\gamma - \alpha)\varphi(t(\alpha), \alpha) + o(\gamma - \alpha).$$

By Lemma 1 $\varphi(t(\alpha), \alpha) < 0$; then $u(t(\alpha), \gamma) < 0$. By the intermediate value theorem we deduce that $t(\gamma) < t(\alpha)$. Using Remark 2 we have $u'(t, \gamma) < 0$ on $(0, t(\gamma))$. Since $\gamma > \alpha$, we have $f(0, \gamma) > f(0, \alpha)$ by (H_1). The lemma is proved.

Let $F(t, u) = \int_0^u f(t, s)\,ds$. We shall also need the following lemma.

Lemma 5. *Assume* (H_1)–(H_3). *Then*:

(i) *For any* $t \in [0, R]$ *there exists* $\beta(t) \in [0, \infty)$ *such that* $f(t, u) < 0$ *for* $u \in (0, \beta(t))$ *and* $f(t, u) > 0$ *for* $u \in (\beta(t), \infty)$. *Moreover* β *is a nondecreasing function.*

(ii) *There exists* $u > 0$ *such that* $F(R, u) \geq 0$.

(iii) *For any* $t \in [0, R]$ *there exists* $\gamma(t) \in [0, \infty)$ *such that* $F(t, u) < 0$ *for* $u \in (0, \gamma(t))$ *and* $F(t, u) > 0$ *for* $u \in (\gamma(t), \infty)$. *Moreover* γ *is a nondecreasing function.*

Proof. (i) is clear. We now prove (ii). Let $v > \beta(R)$ be fixed. Using (i) and (H_1) we have

$$f(R, u) > uf(R, v)/v > 0 \quad \text{for } u > v.$$

Therefore for $u > v$ we can write

$$\begin{aligned} F(R, u) &= \int_0^v f(R, s)\,ds + \int_v^u f(R, s)\,ds \\ &> \int_0^v f(R, s)\,ds + (u^2 - v^2)f(R, v)/2v \end{aligned}$$

and (ii) follows. Now from (H_1) (resp. (H_2)) we deduce that $uf(t, u) > 2F(t, u)$ for $(t, u) \in [0, R] \times (0, \infty)$ (resp. $F_t(t, u) \leq 0$ for $(t, u) \in [0, R] \times [0, \infty)$). Then (iii) is immediate and the lemma is proved.

We conclude this section with the proof of Theorem 2.

(i) Suppose that there exists $\alpha \in (\alpha_0, \infty)$ such that $u(\cdot, \alpha)$ is not admissible and define

$$\alpha_1 = \inf\{\alpha \in (\alpha_0, \infty); u(\cdot, \alpha) \text{ is not admissible}\} < \infty.$$

By Lemma 4 $u(\cdot, \alpha_1)$ is not admissible. Since $\alpha_1 > \alpha_0$, (H_1) implies that $f(0, \alpha_1) > f(0, \alpha_0) > 0$. Therefore we have two cases to consider.

Case 1: $t(\alpha_1) > R$. As before $T_\alpha > R$ for $|\alpha - \alpha_1|$ sufficiently small. The continuity of $u(\cdot, \alpha)$ with respect to the initial value α implies that there exists $\eta > 0$ such that $u(t, \alpha) > 0$ for $t \in [0, R]$ and $\alpha \in (\alpha_1 - \eta, \alpha_1 + \eta)$. Thus we obtain a contradiction with the fact that $u(\cdot, \alpha)$ is admissible for all $\alpha \in [\alpha_0, \alpha_1)$.

Case 2: $t(\alpha_1) \leq R$ and there exists $t_0 \in (0, t(\alpha_1))$ such that $u'(t_0, \alpha_1) > 0$. By Remark 2 $t(\alpha_1) = T_{\alpha_1}$. Then there exists $s_n \in (0, t(\alpha_1))$ such that $s_n \to t(\alpha_1)$ and $u(s_n, \alpha_1) \to \infty$ as $n \to \infty$. Using (H_2) we have

(3.5)
$$\frac{d}{dt}(u'(t, \alpha_1)^2/2 + F(t, u(t, \alpha_1))) = F_t(t, u(t, \alpha_1)) \leq 0 \quad \text{for } t \in [0, t(\alpha_1)).$$

Since $f(0, \alpha_1) > 0$, there exists $\eta > 0$ such that $u'(t, \alpha_1) < 0$ for $t \in (0, \eta)$. From $u'(t_0, \alpha_1) > 0$ we deduce that there exists $t_1 \in (0, t_0)$ such that $u'(t_1, \alpha_1) = 0$ and $u''(t_1, \alpha_1) \geq 0$. Therefore $f(t_1, u(t_1, \alpha_1)) \leq 0$ from which we get, using (i) in Lemma 5, $F(t_1, u(t_1, \alpha_1)) < 0$. Thus $u'(t_1, \alpha_1)^2/2 + F(t_1, u(t_1, \alpha_1)) < 0$. By virtue of (iii) in Lemma 5 we can choose n such that $s_n > t_1$ and $F(R, u(s_n, \alpha_1)) > 0$. Then using (H_2) we have $u'(s_n, \alpha_1)^2/2 + F(s_n, u(s_n, \alpha_1)) \geq F(R, u(s_n, \alpha_1)) > 0$ and we get a contradiction with (3.5).

(ii) Let $\alpha \in [\alpha_0, \infty)$ and $\gamma > \alpha$ with $\gamma - \alpha$ sufficiently small. Using (i) we show that $t(\gamma) < t(\alpha)$ in the same way as in the proof of Lemma 4.

The proof of the theorem is complete.

Remark 4. Let $f \in C^1([0, R] \times [0, \infty))$ be such that $f(t, u) \geq 0$ for $(t, u) \in [0, R] \times [0, \infty)$. Assume (H_2) and

$(\mathrm{H}_1)'$ $\quad uf_u(t, u) > f(t, u) \quad$ for $(t, u) \in [0, R) \times (0, \infty)$.

Then Theorems 1 and 2 still hold. We indicate below the slight modifications.

First we note that $(\mathrm{H}_1)'$ implies $f(t, 0) = 0$ for all $t \in [0, R]$ and that, for all $t \in [0, R)$, $f(t, \cdot)$ is strictly increasing on $[0, \infty)$. Therefore $f(t, u) > 0$ for $t \in [0, R)$ and $u > 0$. Thus $f(0, u(0)) > 0$ in the proof of Theorem 1. In Lemma 3 the case $u'(t(\alpha), \alpha) = 0$ cannot occur by the uniqueness theorem for the initial value problem for ordinary differential equations. In Lemma 5 (i) and (iii) are still valid for $t \in [0, R)$ with $\beta(t) = \gamma(t) = 0$ and (ii) is true for all $u \geq 0$. Actually Lemma 5 is not needed in the present situation. Indeed we note that, in the proof of Theorem 2, Case 2 obviously cannot occur.

4. Examples

4.1. Let $f \in C^1([0, \infty), \mathbb{R})$ be such that $uf'(u) > f(u)$ for $u > 0$. Then Theorem 1 applies. Indeed we note that if $f(u) < 0$ for $u > 0$, then problem (1.1) has no solutions. We give below some examples for which existence is well known (see [10]).

(i) $f(u) = \sum_{j=1}^{k} a_j u^{p_j}$ with $p_j > 1$ and $a_j > 0$ for $j = 1, \ldots, k$;
(ii) $f(u) = u^p/(1+u^s)$ with $p - 1 > s > 0$;
(iii) $f(u) = -u + u^p$ with $p > 1$.

4.2. Let $f \in C^1([0, \infty), \mathbb{R})$ satisfy $f(u) \geq 0$ for $u \geq 0$ and $uf'(u) > f(u)$ for $u > 0$. Let $a \in C^1([0, R])$ be such that $a' \leq 0$ on $[0, R]$, $a(t) > 0$ for $t \in [0, R)$ and $a(R) \geq 0$. Define

$$f(t, u) = a(t)f(u).$$

Then, by Remark 4, Theorem 1 applies.

4.3. Let $a, b \in C^1([0, R])$ be such that $a' \geq 0$, $b' \leq 0$ on $[0, R]$, $a(0) > 0$ and $b(R) > 0$. Define

$$f(t, u) = -a(t)u + b(t)u^p \quad \text{where } p > 1.$$

Then the existence of a solution of (1.1) is known [10]. Theorem 1 gives uniqueness.

4.4. Let $a, b \in C^1([0, R])$ be such that $a', b' \leq 0$ on $[0, R]$ and $a(R)$, $b(R) > 0$. Define

$$f(t, u) = \lambda a(t)u + b(t)u^p$$

with $p > 1$ and $0 \leq \lambda < \lambda_1$ where λ_1 is the smallest eigenvalue of

$$\begin{cases} v''(t) + \mu a(|t|)v(t) = 0, & -R < t < R, \\ v(\pm R) = 0. \end{cases}$$

Then the existence of a solution of (1.1) is well known [10] and Theorem 1 gives uniqueness. If $b(t) > 0$ for $t \in [0, R)$ and $b(R) \geq 0$, then, by Remark 4, Theorem 1 implies uniqueness.

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