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ON THE CARLESON MEASURE CHARACTERIZATION OF BMO FUNCTIONS ON THE UNIT SPHERE

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ABSTRACT. A higher dimensional version of the well-known Carleson measure characterization of BMO functions is given.

1. INTRODUCTION

The main purpose of this paper is to prove a higher dimensional version of the well-known Carleson measure characterization of BMO functions.

Let B denote the unit ball in C^n, $n \geq 1$, and m the $2n$-dimensional Lebesgue measure on B normalized so that $m(B) = 1$, while σ is the normalized surface measure on its boundary S. For the most part we will follow the notation and terminology of Rudin [9].

For $\xi \in S$ and $0 < \delta \leq 2$, put $Q_\delta(\xi) = \{\eta \in S : |1 - \langle \eta, \xi \rangle| < \delta\}$.

The class BMO consists of functions $f \in L^2(\sigma)$ for which

$$||f||^2_{\mathrm{BMO}} = \sup \frac{1}{\sigma(Q)} \int_Q |f(\xi) - f_Q|^2 \, d\sigma(\xi) < \infty,$$

where f_Q denotes the averages of f over Q and the supremum is taken over all $Q = Q_\delta(\xi)$.

A positive Borel measure μ on B is called a Carleson measure if $\mu(B_\delta(\xi)) \leq C\delta^n$, for all $\xi \in S$, $0 < \delta \leq 2$, where $B_\delta(\xi) = \{z \in B : |1 - \langle z, \xi \rangle| < \delta\}$.

Here and elsewhere constants are denoted by C, which may indicate a different constant from one occurrence to the next.

For $f \in C^1(B)$, $Df = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$ denotes the complex gradient of f, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{2n}}\right)$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient of f and let

$$|\nabla_T f(z)|^2 = 2\left(|Df(z)|^2 - |Rf(z)|^2 + \left|D\overline{f}(z)\right|^2 - \left|R\overline{f}(z)\right|^2\right)$$

be the tangential gradient of f. As usual, R denotes the radial derivative $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$.

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For $\psi \in L^1(\sigma)$, $P[\psi]$ denotes the Poisson-Szegő integral defined for $z \in B$ by $P[\psi](z) = \int_S P(z, \xi)\psi(\xi)\, d\sigma(\xi)$, where $P(z, \xi) = \left(\frac{1-|z|^2}{|1-\langle z, \xi\rangle|^2}\right)^n$ is the Poisson-Szegő kernel for B.

Theorem. *Let $f \in L^2(\sigma)$ and $F = P[f]$. Then the following statements are equivalent:*

(a) $f \in \mathrm{BMO}$.

(b) $d\mu(z) = |\nabla F(z)|^2 (1 - |z|^2)\, dm(z)$ *is a Carleson measure.*

(c) $d\nu(z) = |\nabla_T F(z)|^2\, dm(z)$ *is a Carleson measure.*

If $n = 1$, then the Theorem is well known (see, for example, [10]). Note that $d\mu(z) = d\nu(z)$ in this case. See also [2].

Recall that H^2 is a subspace of $L^2(\sigma)$ consisting of functions f such that $P[f]$ is holomorphic in B. If $f \in H^2$, $n > 1$, the equivalence of statements (a) and (b) was proved in [5] and the equivalence (a) $\Longleftrightarrow$ (c) in [3].

2. Preliminaries

As in [9], we say that a function $u \in C^2(B)$ is $\mathscr{M}$-harmonic in B, $u \in \mathscr{M}$, if $\widetilde{\Delta} u(z) = 0$ for every $z \in B$. The operator $\widetilde{\Delta}$ is the invariant Laplacian defined by $\widetilde{\Delta} u(z) = \Delta(u \circ \varphi_z)(0)$, $z \in B$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B taking 0 to z (see [9]).

For $a \in B$ and $0 < r < 1$ let $E_r(a) = \{ z \in B : |\varphi_a(z)| < r \}$. The measure τ defined on B by $d\tau(z) = (1 - |z|^2)^{-n-1}\, dm(z)$ is M-invariant (see [9]).

Lemma 2.1 ([3], [8]). *If $f \in L^2(\sigma)$, then*

$$\int_B G(z)\widetilde{\Delta}|P[f]|^2(z)\, d\tau(z) = \int_S |f(\xi) - P[f](0)|^2\, d\sigma(\xi), \tag{2.1}$$

where $G(z) = \frac{1}{2n}\int_{|z|}^1 t^{1-2n}(1 - t^2)^{n-1}\, dt$, $z \in B \setminus \{0\}$.

Lemma 2.2 ([7]). *Let $F = P[f]$, $f \in L^2(\sigma)$. Then*

$$\begin{aligned}|z|^2\, |\nabla_T F(z)|^2 &= 2\Big((1 - |z|^2)(|RF(z)|^2 + |R\overline{F}(z)|^2) \\ &\qquad + \sum_{i<j} |T_{ij}F(z)|^2 + \sum_{i<j} |T_{ij}\overline{F}(z)|^2\Big),\end{aligned} \tag{2.2}$$

where $T_{ij} = \overline{z}_i \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial z_i}$ *are tangential derivatives.*

Lemma 2.3 ([7]). *Let $1 \le i < j \le n$, $1 \le k \le n$, $0 < r < 1$. There is a constant K such that if $f \in \mathscr{M}$, then*

$$|T_{ij}u_k(w)| \le K \left(1 - |w|^2\right)^{-1/2} \int_{E_r(w)} |u_k(z)|\, d\tau(z), \qquad w \in B, \tag{2.3}$$

where $u_k = \frac{\partial f}{\partial z_k}$ *or* $u_k = \frac{\partial f}{\partial \overline{z}_k}$.

Remark. In [7] analogous estimates for $T_{ij}Rf$ and $T_{ij}\overline{R\overline{f}}$, based on the reproducing formulas which are derived in [1], are obtained. Since the same reproducing formulas holds for the functions u_k (see [1]), the same argument can be applied to derive the estimate (2.3). Similarly, one can conclude that

(2.4)
$$|T_{ij}f(w)| \le K\left(1-|w|^2\right)^{-1/2} \int\limits_{E_r(w)} |f(z)|\,d\tau(z), \qquad w\in B,\ 1\le i<j\le n.$$

Lemma 2.4. *Let* $f \in L^2(\sigma)$ *and* $F = P[f]$. *Then*

$$\int\limits_B G(w)\tilde{\Delta}|F|^2(w)\,d\tau(w) \le C\int\limits_B \left(1-|w|^2\right)^n \tilde{\Delta}|F|^2(w)\,d\tau(w).$$

Proof. For $|z|>1/4$, we have $G(z)\le 16^n(1-|z|^2)^n$.

We may proceed as in the proof of Lemma 2.1 ([7]) to conclude that $|RF(z)|^2$, $|R\overline{F}(z)|^2$, $|T_{ij}F(z)|^2$ and $|T_{ij}\overline{F}(z)|^2$, $1\le i<j\le n$, have $\mathscr{M}$-subharmonic behaviour. Hence, $\tilde{\Delta}|F|^2(z)$ has $\mathscr{M}$-subharmonic behaviour by (2.2). Thus, for $|z|\le 1/4$, we have

$$\tilde{\Delta}|F|^2(z) \le C\int\limits_{E_{1/4}(z)} \tilde{\Delta}|F|^2(w)\,d\tau(w) \le C\int\limits_{\frac{1}{2}B} \tilde{\Delta}|F|^2(w)(1-|w|^2)^n\,d\tau(w),$$

and therefore

$$\int\limits_{|z|\le 1/4} G(z)\tilde{\Delta}|F|^2(z)\,d\tau(z) \le C\int\limits_B (1-|w|^2)^n\tilde{\Delta}|F|^2(w)\,d\tau(w).$$

3. Proof of the Theorem

The proof of Theorem 1 ([5]) shows that if $f\in \mathrm{BMO}$, then $d\mu(z)$ is a Carleson measure.

(b) $\Longrightarrow$ (c) It is easy to see that if $d\mu(z)$ is a Carleson measure, then $(1-|z|^2)|RF(z)|^2dm(z)$ and $(1-|z|^2)|R\overline{F}(z)|^2dm(z)$ are Carleson measures. Hence, by Lemma 2.2, to show that $d\nu(z)$ is a Carleson measure it suffices to show that $|T_{ij}F(z)|^2\,dm(z)$ and $\left|T_{ij}\overline{F}(z)\right|^2 dm(z)$ are Carleson measures for all $1\le i<j\le n$.

We will show that $|T_{ij}F(z)|^2\,dm(z)$ is a Carleson measure. Analogously, we may prove that $\left|T_{ij}\overline{F}(z)\right|^2 dm(z)$ is a Carleson measure.

An integration by parts shows that

$$F(z) = \int_0^1 \left\{RF(tz) + \overline{R\overline{F}}(tz) + F(tz)\right\} dt.$$

From this we conclude that it is sufficient to prove that

$$\left[\int_0^1 \left|T_{ij}\frac{\partial F}{\partial z_k}(tz)\right| dt\right]^2 dm(z),$$

$$\left[\int_0^1 \left|T_{ij}\frac{\partial F}{\partial \overline{z}_k}(tz)\right| dt\right]^2 dm(z) \quad\text{and}\quad \left[\int_0^1 |T_{ij}F(tz)|\,dt\right]^2 dm(z)$$

are Carleson measures for all $1\le k\le n$. Let $u_k = \frac{\partial F}{\partial z_k}$ or $u_k = \frac{\partial F}{\partial \overline{z}_k}$.

Using Lemma 2.3 and the fact that $1-|w|^2 \cong |1-\langle z, w\rangle| \cong 1-|z|^2$, for $w \in E_r(z)$, we see that for any $s>0$

$$\int_0^1 |T_{ij}u_k(tw)|\,dt \le C\int_0^1\left[\int_{E_r(tw)} \frac{(1-|z|^2)^s\,|u_k(z)|}{|1-t\langle w, z\rangle|^{n+s+3/2}}\,dm(z)\right]dt$$
$$\le C\int_0^1\left[\int_B \frac{(1-|z|^2)^s}{|1-t\langle w, z\rangle|^{n+s+3/2}}\,|u_k(z)|\,dm(z)\right]dt$$
$$\le C\int_B \frac{(1-|z|^2)^s\,|u_k(z)|}{|1-\langle w, z\rangle|^{n+s+1/2}}\,dm(z).$$

Now we proceed as in [6]. Let $\xi \in S$ and $\delta>0$. By standard estimates (see [9], p. 17)

$$\int_B \frac{(1-|z|^2)^{s-1/4}}{|1-\langle w, z\rangle|^{n+s+5/4}}\,dm(z) \le C\left(1-|w|^2\right)^{-1/2}.$$

Thus by Hölders inequality

$$\int_{B_\delta(\xi)}\left[\int_0^1 |T_{ij}u_k(tw)|\,dt\right]^2 dm(w)$$
$$\le C\int_{B_\delta(\xi)}\left[\int_B \frac{(1-|z|^2)^s\,|u_k(z)|}{|1-\langle z, w\rangle|^{n+s+1/2}}\,dm(z)\right]^2 dm(w)$$
$$\le C\int_{B_\delta(\xi)}\left[\int_B \frac{|u_k(z)|^2\,(1-|z|^2)^{s+1/4}}{|1-\langle z, w\rangle|^{n+s-1/4}}\,dm(z)\right.$$
$$\left.\times\int_B \frac{(1-|z|^2)^{s-1/4}}{|1-\langle z, w\rangle|^{n+s+5/4}}\,dm(z)\right]dm(w)$$
$$\le C\int_{B_\delta(\xi)}\left[\int_B \frac{|u_k(z)|^2\,(1-|z|^2)^{s+1/4}}{|1-\langle z, w\rangle|^{n+s-1/4}}\,dm(z)\right]\left(1-|w|^2\right)^{-1/2} dm(w)$$
$$= C\int_{B_\delta(\xi)}\left[\int_{B_{2\delta}(\xi)} \frac{|u_k(z)|^2\,(1-|z|^2)^{s+1/4}}{|1-\langle z, w\rangle|^{n+s-1/4}}\,dm(z)\right]\left(1-|w|^2\right)^{-1/2} dm(w)$$
$$+ C\int_{B_\delta(\xi)}\left[\sum_j\int_{A_j} \frac{|u_k(z)|^2\,(1-|z|^2)^{s+1/4}}{|1-\langle z, w\rangle|^{n+s-1/4}}\,dm(z)\right]\left(1-|w|^2\right)^{-1/2} dm(w)$$
$$= I_1 + I_2,$$

where $A_j = \{z \in B : 2^j\delta \le |1-\langle z, \xi\rangle| < 2^{j+1}\delta\}$, $j = 1, 2, \ldots$. As above we have

$$\int_B \frac{(1-|w|^2)^{-1/2}}{|1-\langle z, w\rangle|^{n+s-1/4}}\,dm(w) \le C\left(1-|z|^2\right)^{3/4-s}$$

(we may suppose that $s>3/4$).

Hence by Fubini's theorem

$$I_1 \leq C \int_{B_{2\delta}(\xi)} |u_k(z)|^2 \left(1-|z|^2\right)^{s+1/4} \left[\int_B \frac{\left(1-|w|^2\right)^{-1/2}}{|1-\langle z, w\rangle|^{n+s-1/4}} dm(w)\right] dm(z)$$
$$\leq C \int_{B_{2\delta}(\xi)} \left(1-|z|^2\right) |u_k(z)|^2\, dm(z) \leq C\delta^n,$$

since, from the assumption that $\left(1-|z|^2\right)|\nabla F(z)|^2 dm(z)$ is a Carleson measure, it follows easily that $\left(1-|z|^2\right)|u_k(z)|^2\, dm(z)$ is a Carleson measure for all $1 \leq k \leq n$.

Notice that if $w \in B_\delta(\xi)$ and $z \in A_j$ we have

$$|1-\langle w, z\rangle|^{1/2} \geq |1-\langle z, \xi\rangle|^{1/2} - |1-\langle w, \xi\rangle|^{1/2} \geq \frac{1}{2}\left(\sqrt{2}-1\right) 2^{j/2}\delta^{1/2}.$$

Hence

$$I_2 \leq C \sum_j \left(2^j\delta\right)^{-n-1/2} \int_{A_j} |u_k(z)|^2 \left(1-|z|^2\right) dm(z) \int_{B_\delta(\xi)} \left(1-|w|^2\right)^{-1/2} dm(w).$$

From the estimate $\sigma(\{\eta \in S : |1-r\langle \eta, \xi\rangle| < \delta\}) \leq C\delta^n$, $0 \leq r < 1$ (see [9], p.67), it follows easily that $\int_{B_\delta(\xi)} (1-|w|^2)^{-1/2}\, dm(w) \leq C\delta^{n+1/2}$. Using this and again the fact that $|u_k(z)|^2(1-|z|^2)\, dm(z)$ is a Carleson measure we get $I_2 \leq C\delta^n$.

Now we will show that $(1-|w|^2)|F(w)|^2 dm(w)$ is also a Carleson measure. Without loss of generality we may assume that $F(0) = 0$. Let $s > 1$ and $0 < r < 1$. Using Lemma 3.1 in [7] we find that

$$|F(w)|^2 = \left|\int_0^1 \frac{d}{dt} F(tw)\, dt\right|^2 \leq \int_0^1 |\nabla F(tw)|^2 dt$$
$$\leq C \int_0^1 \left(\int_{E_r(tw)} \frac{|\nabla F(z)|^2 (1-|z|^2)^s dm(z)}{|1-\langle z, tw\rangle|^{n+1+s}} \right) dt$$
$$\leq C \int_0^1 \left(\int_B \frac{|\nabla F(z)|^2 (1-|z|^2)^s dm(z)}{|1-t\langle z, w\rangle|^{n+1+s}} \right) dt$$
$$\leq C \int_B \frac{|\nabla F(z)|^2 (1-|z|^2)^s dm(z)}{|1-\langle z, w\rangle|^{n+1+s}}.$$

Here we have used the fact that $|1-\langle z, tw\rangle| \cong 1-|z|^2$, for $z \in E_r(tw)$, and the simple estimate $|1-t\langle z, w\rangle|^{-1} \leq 2|1-\langle z, w\rangle|^{-1}$, $0 < t < 1$, $z, w \in B$. (In fact, by an integration we can get a better estimate.)

Thus, for $\xi \in S$ and $\delta > 0$ we have

$$\begin{aligned}\int_{B_\delta(\xi)} &(1-|w|^2)|F(w)|^2 dm(w)\\ &\le C\int_{B_\delta(\xi)}(1-|w|^2)\left(\int_{B_{2\delta}(\xi)}\frac{|\nabla F(z)|^2(1-|z|^2)^s dm(z)}{|1-\langle z, w\rangle|^{n+1+s}}\right)dm(w)\\ &+C\int_{B_\delta(\xi)}\left(\sum_j\int_{A_j}\frac{|\nabla F(z)|^2(1-|z|^2)^s dm(z)}{|1-\langle z, w\rangle|^{n+1+s}}\right)(1-|w|^2)dm(w),\end{aligned}$$

where $A_j = \{z \in B : 2^j\delta \le |1-\langle z, \xi\rangle| < 2^{j+1}\delta\}$, $j = 1, 2, \dots$.

Since $(1-|z|^2)|\nabla F(z)|^2 dm(z)$ is a Carleson measure, as above, we find that $\int_{B_\delta(\xi)}(1-|w|^2)|F(w)|^2dm(w) \le C\delta^n$.

Using this and the estimate (2.4) we conclude that $(\int_0^1 |T_{ij}F(tz)|^2dt)^2dm(z)$ is a Carleson measure for all $1 \le i, j \le n$.

(c) $\Rightarrow$ (a) Assume that $d\nu(z)$ is a Carleson measure. Then

$$\sup_{z\in B}\int_B \frac{(1-|z|^2)^n}{|1-\langle w, z\rangle|^{2n}}d\nu(w) < \infty$$

(see [3]). To show that $f \in \mathrm{BMO}$ it suffices to show that the Garcia norm of f,

$$\|f\|_G^2 = \sup_{z\in B}\int_S |f(\xi) - P[f](z)|^2 P(z, \xi)\, d\sigma(\xi),$$

is finite.

By (2.1) and Lemma 2.4 we have

$$\|f\|_G^2 \le C\sup_{z\in B}\int_B (1-|w|^2)^n \tilde{\Delta}\,|P\,[f\circ\varphi_z]|^2\,(w)\,d\tau(w).$$

Note by the M-invariance of P, $\tilde{\Delta}$ and τ that the last integral is equal to $\int_B \left(1-|\varphi_z(w)|^2\right)^n \tilde{\Delta}|P[f]|^2(w)\,d\tau(w)$, which is in turn equal to

$$\int_B \frac{(1-|z|^2)^n(1-|w|^2)^n}{|1-\langle z, w\rangle|^{2n}}\tilde{\Delta}|P[f]|^2(w)d\tau(w),$$

by Theorem 2.2.2 ([9], p. 26).

Using the formula for the invariant Laplacian given in Theorem 4.1.3 of Rudin's book [9] and the fact that $P[f]$ is an $\mathscr{M}$-harmonic function we find that $\tilde{\Delta}|P[f]|^2(w) = 2(1-|w|^2)|\nabla_T P[f](w)|^2$. Thus,

$$\|f\|_G^2 \le C\sup_{z\in B}\int_B \frac{(1-|z|^2)^n}{|1-\langle z, w\rangle|^{2n}}d\nu(w) < \infty.$$

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