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A REFLEXIVE ADMISSIBLE TOPOLOGICAL GROUP MUST BE LOCALLY COMPACT

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ABSTRACT. Let G be a reflexive topological group, and $G^\frown$ its group of characters, endowed with the compact open topology. We prove that the evaluation mapping from $G^\frown \times G$ into the torus T is continuous if and only if G is locally compact. This is an analogue of a well-known theorem of Arens on admissible topologies on $C(X)$.

DEFINITIONS AND REMARKS

Let X, Y be topological spaces, and let Z be a subset of Y^X. A topology on Z is said to be admissible if the evaluation mapping from the product $Z \times X$ into Y, defined by $w(f, x) = f(x)$, is continuous.

Let $(S, V) = \{f \in Z;\ f(S) \subseteq V\}$. The family $\{(S, V)\}$, where S runs over the collection of all compact subsets of X and V runs over a basis of open sets in Y, is a subbase for the compact open topology on Z. An admissible topology on Z must be finer than the compact open topology [8]. A result of Arens states that the existence of a coarsest admissible topology for the class of real continuous functions on a completely regular space X is equivalent to X being locally compact [1].

In this paper we are interested in reflexive topological groups. Answering in the negative a question of Megrelishvili [7], we will see that the evaluation map for those groups need not be continuous. In fact, as specified in the theorem, the continuity of the evaluation characterizes locally compact Hausdorff abelian groups among reflexive groups. We prove this result by means of convergence spaces. For an account of theory of convergence spaces the reader is referred to [3, 4]. We only give here needed definitions.

Let G be a Hausdorff topological abelian group, and let ΓG be the set of all continuous homomorphisms from G into the unidimensional torus T. If addition is defined pointwise in ΓG, it becomes an abelian group. The *continuous*

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convergence structure Λ on ΓG is defined as follows:

A filter $\mathscr{F}$ in ΓG is said to converge to $f \in \Gamma G$ in Λ if for every filter $\mathscr{U}$ in G, which converges to an element $x \in G$, the filter of basis $w(\mathscr{F} \times \mathscr{U}) := \{w(F \times U);\ F \in \mathscr{F} \text{ and } U \in \mathscr{U}\}$ converges to $f(x)$ in T.

The continuous convergence structure is compatible with the addition in ΓG, so that the group ΓG endowed with Λ is a *convergence group*, in the sequel denoted by $\Gamma_c G$.

We shall denote by $G^\wedge$ the set ΓG endowed with the compact open topology, τ_c. The group G is said to be *reflexive* if the canonical embedding α_G: $G \to G^{\wedge\wedge}$, where $G^{\wedge\wedge} = (G^\wedge)^\wedge$, is a topological isomorphism.

A theorem of Arens type

The purpose of this article is to prove the following.

Theorem. *If G is a reflexive topological abelian group, then the evaluation mapping w: $G^\wedge \times G \to T$ is continuous if and only if G is locally compact.*

Before the proof, we state two auxiliary propositions.

Proposition 1. *Let G be a topological abelian group. If α_G: $G \to G^{\wedge\wedge}$ is continuous, then $\Gamma_c G$ is locally compact in the sense of convergence, i.e. every Λ-convergent filter in ΓG has a compact member.*

Proof. Let $\mathscr{F}$ be a Λ-convergent filter in ΓG. Without loss of generality, suppose $\mathscr{F} \xrightarrow{\Lambda} 0$. If $\mathscr{N}$ denotes the zero neighborhood system in G, then $w(\mathscr{F} \times \mathscr{N}) \to 0$ in T; thus we can find $F \in \mathscr{F}$ and $N \in \mathscr{N}$ such that $w(F \times N) \subseteq [-1/4, 1/4]$ (here we identify the points of T with elements of $(-1/2, 1/2])$.

Let $N^\circ := \{\phi \in \Gamma G;\ \phi(N) \subseteq [-1/4, 1/4]\}$. This set, being the polar of a neighborhood of zero, is τ_c-compact ([2], (1.5)). It is also equicontinuous. In fact, if V denotes a zero neighborhood in T, by the continuity of α_G, a zero neighborhood M in $\mathscr{N}$ can be determined so that $\alpha_G(M) \subseteq (N^\circ, V)$. Thus, $\psi(x) \in V$, for every $\psi \in N^\circ$ and every $x \in M$.

We claim that N° is Λ-compact. In order to prove this, take an ultrafilter $\mathscr{U}$ in N° and suppose $\mathscr{U} \xrightarrow{\tau_c} \varphi$. By the last assumption and the equicontinuity of N°, $w(\mathscr{U} \times (x + \mathscr{N})) \to \varphi(x)$ for every x in G, which implies that $\mathscr{U} \xrightarrow{\Lambda} \varphi$.

Finally, since $N^\circ \supseteq F$, it belongs to $\mathscr{F}$; therefore $\Gamma_c G$ is locally compact.

Proposition 2. *If G is a topological Hausdorff abelian group, the following statements are equivalent*:

(i) *The evaluation w: $G^\wedge \times G \to T$ is continuous.*
(ii) *The continuous convergence structure Λ defined in ΓG coincides with the convergence structure of the compact open topology τ_c.*

Proof. (i) $\Rightarrow$ (ii) Clearly if w: $G^\wedge \times G \to T$ is continuous, every τ_c-convergent filter in ΓG is Λ-convergent. The converse holds without the assumption (i). In fact, take a filter $\mathscr{F}$ in ΓG which is not τ_c-convergent to the null character. Let $S \subseteq G$ be a compact subset of G and V a neighborhood of zero in T, such that $(S, V) \notin \mathscr{F}$. For every $F \in \mathscr{F}$, we can find ψ_F in F and x_F in S such that $\psi_F(x_F) \notin V$.

The net $\mathscr{S} = \{x_F, F \in \mathscr{F}\}$, where $\mathscr{F}$ is directed by $\supseteq$, has a convergent subnet, say $\mathscr{R} \to x \in S$. The filter associated to $\mathscr{R}$, $\mathscr{G}_{\mathscr{R}}$, also converges to x [6]. It can be easily seen that $w(\mathscr{F} \times \mathscr{G}_{\mathscr{R}}) \nrightarrow 0$, thus $\mathscr{F}$ does not converge in Λ to the null character.

(ii) $\Rightarrow$ (i) is obvious.

Proof of the theorem. If the evaluation is continuous, $G^\wedge$ may be identified with $\Gamma_c G$ by Proposition 2 and it is locally compact by Proposition 1. Its dual $G^{\wedge\wedge}$ is locally compact [8], and so is G, which is topologically isomorphic to $G^{\wedge\wedge}$.

Conversely, let V be a neighborhood of zero in T. If K denotes a compact neighborhood of zero in G, the inclusion $w((K, V) \times K) \subseteq V$ proves the continuity of w.

Remark 1. The assumption of reflexivity cannot be dropped in the theorem, as the following examples show. The spaces L_p, for $0 < p < 1$, considered in their additive structure, are topological groups without non-trivial characters [5]. Thus, for any such group $G^\wedge = \{0\}$ and the evaluation is a constant map, thereby continuous. The same happens with the so-called exotic groups. (For an account on those, see [2].)

Remark 2. The existence of many classes of reflexive, non-locally compact groups is a well-known fact [2]. This theorem could perhaps explain why the class of locally compact Hausdorff abelian groups fits best the Pontryagin-van Kampen duality theorem.

Corollary. *If α_G is a topological embedding from G into $G^{\wedge\wedge}$, the following facts are equivalent*:

(i) w: $G^\wedge \times G \to T$ *is continuous,*

(ii) $G^\wedge$ *is locally compact.*

If (i) *and* (ii) *hold, then $\alpha_G(G)$ is dense in the locally compact group $G^{\wedge\wedge}$.*

Proof. The equivalence follows easily from Propositions 1 and 2. The fact that $\alpha_G(G)$ is a subgroup of $G^{\wedge\wedge}$ that separates points of $G^\wedge$, together with Proposition 31 of [8], proves the last part.

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