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A GEOMETRICAL CHARACTERIZATION OF ALGEBRAIC VARIETIES OF $\mathbb{C}^2$

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ABSTRACT. We prove that a closed subset $V \neq \mathbb{C}^2$ is an algebraic variety if all its horizontal sections and its vertical sections are finite or a complex line and if $\mathbb{C}^2 \backslash V$ is an open set of holomorphy (equivalently pseudoconvex). This result has important consequences in the theory of the socle for Jordan-Banach algebras.

Suppose that $V \neq \mathbb{C}^2$ is an algebraic variety of $\mathbb{C}^2$. If we intersect V with horizontal lines $\mathbb{C} \times \{t\}$ or vertical lines $\{s\} \times \mathbb{C}$ it is obvious that the intersections are either finite or all these lines. Is the converse true? Certainly not, as the example $V = \{(\lambda, \overline{\lambda}) : \lambda \in \mathbb{C}\}$ shows. But if V is algebraic it is defined by an equation $G(s, t) = 0$, where G is a polynomial. Then $H(s, t) = \frac{1}{G(s,t)}$ is holomorphic on $\mathbb{C}^2 \backslash V$ and certainly has no extension through every point of V, so this means that $\mathbb{C}^2 \backslash V$ is an open set of holomorphy. We shall see in the main theorem that this extra condition with the two previous ones on finite sections are enough to characterize algebraic varieties.

It is well known that an open set $\Omega \subset \mathbb{C}^n$ is an open set of holomorphy if and only if it is pseudoconvex (this geometric condition means that $-\operatorname{Log}\operatorname{dist}(z, \partial\Omega)$ is plurisubharmonic on Ω). This was proved by K. Oka in 1942 for $n = 2$ and for $n \geq 2$ by K. Oka, F. Norguet, and H. J. Bremermann in 1952–1954. See a standard textbook on functions of several complex variables, for instance [H] or the Appendix of [A2].

For the definitions and properties related to analytic multifunctions we refer to [A2, 4].

Lemma 1. *Let $V \neq \mathbb{C}^2$ be the complement of an open set of holomorphy of $\mathbb{C}^2$, and let U be a domain of $\mathbb{C}$. Suppose that for $s \in U$ the sections $(\{s\} \times \mathbb{C}) \cap V = K(s)$ are compact and nonempty. Then K is an analytic multifunction on U. If, moreover, these sections are finite for all $s \in U$, there exist an integer m and a closed discrete subset E of U such that $\#K(s) = m$ on $U \backslash E$ and $\#K(s) < m$ on E. Then there exist m holomorphic functions $a_0, \ldots, a_{m-1}$ on U such that $t \in K(s)$, for $s \in U$, is equivalent to $t^m + a_{m-1}(s)t^{m-1} + \cdots + a_0(s) = 0$.*

Proof. We know that $\mathbb{C}^2 \backslash V$ is an open set of holomorphy, so it is pseudoconvex and locally pseudoconvex. This implies that the complement of the graph of

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K on U is pseudoconvex. By Słodkowski's Theorem ([A2], Theorem 7.1.10), K is analytic multivalued. The rest of the lemma comes from the Scarcity Theorem for elements with finite values ([A2], Theorem 7.1.7) and the argument of Theorem 7.3.5 in [A2]. □

Lemma 2 (Casorati-Weierstrass Theorem for analytic multifunctions). *Denote by Δ an open set containing a finite number of points and by Δ' the punctured open set obtained from Δ by deleting these points. Suppose that K is an analytic multifunction defined on Δ', and let $\overline{B}(\alpha, r)$ be a disk of center α and radius $r > 0$ such that $K(s) \cap \overline{B}(\alpha, r) = \varnothing$ for $s \in \Delta'$. Then the multifunction*

$$M(s) = \left\{ \frac{1}{t - \alpha} : t \in K(s) \right\}$$

has an analytic extension to all Δ.

Proof. By Localization Principle ([A2], Theorem 7.1.5) it is clear that M is analytic on Δ'. From the hypothesis it is uniformly bounded on Δ', so it is possible to define $M(a) = \limsup_{s \to a, s \neq a} M(s)$ at all points of the deleted finite set. Using Słodkowski's Theorem ([A2], Theorem 7.1.10(ii)) and the fact that a bounded subharmonic function on Δ' admits a subharmonic extension to Δ (see [HK], Theorem 5.18) we get the result. Lemma 2 is also a consequence of the slightly more general Theorem 6.4 of [R]. With the notation of this last theorem we have $G' = \Delta'$, P is the finite set and $\bigcap_{0<|s-a|<1/n} K(s)$ avoids $\overline{B}(\alpha, r)$ which is nonpolar because it has capacity r. □

This lemma says that if K is not obtained by a Möbius transform from an analytic multifunction on Δ, then $\bigcup_{0<|s-a|<r} K(s)$ is dense in the complex plane for every $r < 1$, where a is in the deleted finite set. In some sense it says that K has essential singularities on the finite set.

The next result is a very famous theorem on the distribution of values of entire functions of two complex variables, proved by Matasugu Tsuji in 1943.

Lemma 3 (Tsuji's Theorem). *Let $G(s, t)$ be an entire function on $\mathbb{C}^2$ which is not of the form $G(s, t) = e^{H(s,t)}$, with H entire on $\mathbb{C}^2$. Then there exists a G_δ-polar subset E of $\mathbb{C}$ such that for $t \notin E$ the equation $G(s, t) = 0$ has a solution. Moreover, if G is not algebroid in s (that is, there are no integers n and no entire functions on $\mathbb{C}$, denoted by $\beta_0, \dots, \beta_n$, such that $G(s, t) = \beta_n(t)s^n + \beta_{n-1}(t)s^{n-1} + \cdots + \beta_0(t)$), then there exists a G_δ-polar subset F of $\mathbb{C}$ such that for $t \notin F$ the equation $G(s, t) = 0$ has an infinite number of solutions.*

Proof. The standard and difficult proof using conformal mapping is given in [T, Theorem VII.37, pp. 329–331]. A more elementary proof based on spectral theory is given in [A1, Theorem 2.15, p. 42]. An elementary proof using analytic multifunctions and pseudoconvex sets is given in [A2, Theorem 7.3.11, p. 169]. □

Using only part (e) of the proof of the main theorem and Tsuji's Theorem it is possible to obtain the following corollary.

Corollary. *Let $m(s) = (a_{ij}(s))$ be a family of $n \times n$ matrices whose entries are entire functions. Then $\bigcup_{s\in\mathbb{C}} Sp\, m(s)$ avoids at most $2n - 1$ points. Suppose that* *for every* $\alpha \in \mathbb{C}$ *the set* $\{s : \alpha \in Spm(s)\}$ *is finite. Then the* n *symmetric functions of the eigenvalues of* $m(s)$, *denoted by* $\sigma_1(s) = trm(s), \sigma_2(s), \dots, \sigma_n(s) = \det m(s)$, *are polynomials.*

For the first part see [Au2, Lemma 7.3.4]. The a_{ij} are not necessarily polynomials as the example $a_{11}(s) = s, a_{12}(s) = e^s, a_{21}(s) = e^{-s}, a_{22}(s) = -s$ shows.

This corollary is in fact a consequence of the more general result given now.

Theorem. *Let* $V \neq \mathbb{C}^2$ *be a closed subset of* $\mathbb{C}^2$ *satisfying the following properties*:

(i) $\mathbb{C}^2 \backslash V$ *is an open set of holomorphy of* $\mathbb{C}^2$,

(ii) *for* $s \in \mathbb{C}$, *either the vertical section* $(\{s\} \times \mathbb{C}) \cap V$ *is finite or* $\{s\} \times \mathbb{C} \subset V$,

(iii) *for* $t \in \mathbb{C}$, *either the horizontal section* $(\mathbb{C} \times \{t\}) \cap V$ *is finite or* $\mathbb{C} \times \{t\} \subset V$.

Then there exist two integers m, n *and two closed discrete subsets* E, F *of* $\mathbb{C}$ *such that* $\#((\{s\} \times \mathbb{C}) \cap V) = m$, *for* $s \notin E$, *and* $\#((\mathbb{C} \times \{t\}) \cap V) = n$, *for* $t \notin F$. *Moreover,* V *is an algebraic variety of* $\mathbb{C}^2$ *of degree* $\leq m + n$.

Proof. (a) Replacing V by $V' = V \cup (\{\alpha\} \times \mathbb{C}) \cup (\mathbb{C} \times \{\beta\})$, where α, β are two fixed complex numbers, we can suppose that all the vertical and horizontal sections are nonempty. It is obvious that V' satisfies (ii) and (iii). To prove that V' satisfies (i) we use a standard theorem on open sets of holomorphy (see [A2, Theorem A.2.5], for instance). Using twice this theorem, first with $D = \mathbb{C}^2 \backslash V$, $u(s, t) = s$, $D' = \mathbb{C} \backslash \{\alpha\}$ and then with $D = (\mathbb{C}^2 \backslash V) \backslash (\{\alpha\} \times \mathbb{C})$, $u(s, t) = t$, $D' = \mathbb{C} \backslash \{\beta\}$, we conclude that $\mathbb{C}^2 \backslash V'$ is an open set of holomorphy. So in the rest of the argument we shall suppose that V contains at least one vertical and one horizontal line.

(b) Because $V \neq \mathbb{C}^2$ the conditions (ii) and (iii) imply that V contains a finite number k of vertical lines and a finite number l of horizontal lines. We denote by $s_1, \dots, s_k$ the feet of these vertical lines and by $t_1, \dots, t_l$ the feet of these horizontal lines. If all the horizontal sections are always equal to $\{s_1, \dots, s_k\}$ or to some horizontal lines, V is the union of k vertical lines and l horizontal lines, so it is an algebraic variety of degree $k + l$. The same is true if the vertical sections are always equal to $\{t_1, \dots, t_l\}$ or to some vertical line. So we now suppose that we are not in this case. For $s \neq s_1, \dots, s_k$ we introduce the multifunction

$$K(s) = \{t : (s, t) \in V\} \neq \varnothing.$$

By Lemma 1, K is algebroid on $U = \mathbb{C} \backslash \{s_1, \dots, s_k\}$ of degree $m > k$, and there exist m functions $a_1, \dots, a_{m-1}$ which are holomorphic on this open set such that

$$(s, t) \in V \Leftrightarrow t^m + a_{m-1}(s) t^{m-1} + \cdots + a_0(s) = 0 \tag{1}$$

for $s \neq s_1, \dots, s_k$. We can take for E the union of the branching points of the algebroid function with $\{s_1, \dots, s_k\}$. If $k = 0$, then $a_0, \dots, a_{m-1}$ are entire, so we suppose in the following argument that $k \geq 1$.

(c) We now prove that $a_0, \dots, a_{m-1}$ are meromorphic on the complex plane; consequently they will be quotients of entire functions ([R, Theorem 15.12, p. 327]). We introduce for $t \neq t_1, \dots, t_l$ the multifunction

$$L(t) = \{s : (s, t) \in V\} \neq \varnothing.$$

The argument of part (b) done with L proves the existence of an integer $n > k$ and a closed discrete set E' containing $\{t_1, \dots, t_l\}$ such that $\#L(t) = n$ for $t \notin E'$. Translating V if necessary, without loss of generality we may suppose that $0 \notin E'$ in which case $L(0)$ contains $\{s_1, \dots, s_k\}$ and $n - k \geq 1$ other points. We choose $\delta > 0$ such that $\overline{B}(s_i, \delta) \cap L(0) = \{s_i\}$ $(1 \leq i \leq k)$, where $\overline{B}(s_i, \delta)$ denotes the closed disk centered at s_i of radius δ. By continuity of L on $\mathbb{C}\backslash E'$ and the fact that $\#L(t) = n$ for $t \notin E'$ we conclude that there exists $\varepsilon > 0$ such that $|t| < \varepsilon$ implies $L(t) \cap (\overline{B}(s_1, \delta) \cup \cdots \cup \overline{B}(s_k, \delta)) = \{s_1, \dots, s_k\}$. Hence $K(s) \cap \overline{B}(0, \varepsilon) = \varnothing$ for $0 < |s - s_1| < \delta, \dots, 0 < |s - s_k| < \delta$. If we introduce

$$M(s) = \left\{ \frac{1}{t} : t \in K(s) \right\}, \qquad s \in \Delta',$$

where Δ' is the punctured domain $\Delta \backslash \{s_1, \dots, s_k\}$ and $\Delta = B(s_1, \delta) \cup \cdots \cup B(s_k, \delta)$, by Lemma 2 we know that M has an analytic multivalued extension to Δ. But for $s \in \Delta'$ we have by (1):

$$u \in M(s) \Leftrightarrow \frac{1}{a_0(s)} + \frac{a_{m-1}(s)}{a_0(s)} u + \cdots + u^m = 0, \tag{2}$$

so because M extends analytically to all Δ, this implies that $\frac{1}{a_0}, \frac{a_{m-1}}{a_0}, \dots, \frac{a_1}{a_0}$ extends analytically to Δ, and hence $a_0, \dots, a_{m-1}$ are meromorphic on all the plane.

(d) Consequently there exist $\alpha_0, \dots, \alpha_m$ entire such that

$$(s, t) \in V \Leftrightarrow G(s, t) = 0, \tag{3}$$

where $G(s, t) = \alpha_m(s)t^m + \alpha_{m-1}(s)t^{m-1} + \cdots + \alpha_0(s)$.

By Lemma 3 if G is not algebroid in s, the equation $G(s, t) = 0$ has an infinite number of solutions in s, for t fixed outside of a polar set F, and this would mean that V contains an infinite number of horizontal lines, which is absurd. So there exist $\beta_0, \dots, \beta_n$ entire such that

$$G(s, t) = \beta_n(t)s^n + \beta_{n-1}(t)s^{n-1} + \cdots + \beta_0(t). \tag{4}$$

(e) Taking the $(m + 1)$-derivative of G in t, which is zero by (3), by (4) we get:

$$\beta_n^{(m+1)}(t)s^n + \beta_{n-1}^{(m+1)}(t)s^{n-1} + \cdots + \beta_0^{(m+1)}(t) = 0, \tag{5}$$

for every s and t. We now take $n + 1$ different values of s, denoted by $\sigma_1, \dots, \sigma_{n+1}$ in formula (5), and we obtain a homogeneous linear system of $n+1$ equations with $n+1$ unknowns $x_0 = \beta_0^{(m+1)}(t)$, $x_1 = \beta_1^{(m+1)}(t), \dots, x_n = \beta_n^{(m+1)}(t)$, whose determinant is a Vandermonde determinant

$$\prod_{1 \leq i < j \leq n+1} (\sigma_i - \sigma_j) = \begin{vmatrix} \sigma_1^n & \sigma_1^{n-1} & \cdots & 1 \\ \vdots & & & \\ \sigma_{n+1}^n & \sigma_{n+1}^{n-1} & \cdots & 1 \end{vmatrix} \neq 0. \tag{6}$$

Consequently $\beta_n^{(m+1)} \equiv \beta_{n-1}^{(m+1)} \equiv \cdots \equiv \beta_0^{(m+1)} \equiv 0$ on $\mathbb{C}$, so the β_i are polynomials of degree at most m. Hence V is an algebraic variety of degree at most $m + n$. $\square$

This theorem was invented to prove very interesting results on the socle in Jordan-Banach algebras. Denoting by $\mathscr{F}_n$ the set of elements of a Jordan-Banach algebra with rank less than or equal to n (for the convenient definition see [A3, §5] or [A4, §8.5]), from the previous theorem we conclude that $a \in \mathscr{F}_m$ and $b \in \mathscr{F}_n$ imply $a + b \in \mathscr{F}_{m+n}$ and consequently the socle of the Jordan-Banach algebra coincides with the union of all the $\mathscr{F}_n$. The details of these results will appear in [A5].

We thank very much the referee of this paper who suggested other possible proofs of the main theorem (which are less elementary). For instance the following one.

If $f(z) = u(x, y) + iv(x, y)$ is holomorphic on domain D of $\mathbb{C}$, then the area differential on the graph of f in $\mathbb{R}^4$ is given by $\|h_1 \wedge h_2\| \, dx \, dy$, where $h_1 = (1, 0, u'_x, v'_x)$, $h_2 = (0, 1, u'_y, v'_y)$. Consequently by Cauchy-Riemann equations we get $\|h_1 \wedge h_2\| \leq \|h_1\| \, \|h_2\| = 1 + |f'(z)|^2$. If f is a conformal mapping this implies that

$$\int_G \|h_1 \wedge h_2\| \, dx \, dy \leq \mathrm{Area}(G) + \mathrm{Area}(G') \tag{7}$$

where $G' = f(G)$. By Lemma 1 we know that V is an m-sheeted cover over the s-plane and an n-sheeted cover over the t-plane. So it is not difficult to conclude from (7) that for B_r denoting the ball of radius r centered at the origin we have

$$\mathrm{Area}(V \cap B_r) \leq \pi(m + n)r^2 . \tag{8}$$

Then from Stoll's Theorem (Theorem D of [S]) or Bishop's Theorem (Theorem F of [S]) we conclude that V is an algebraic variety.

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