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GENERALIZATIONS OF M-GROUPS

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ABSTRACT. In this note we prove some generalizations of Taketa's theorem on solvability of M-groups.

Let $\operatorname{Irr}(G)$ denote the set of all complex irreducible characters of a finite group G (only finite groups are considered in this note), $\operatorname{Irr}_1(G)$ the set of all nonlinear characters in $\operatorname{Irr}(G)$, $\operatorname{Lin}(G)$ the set of all linear characters of G, and $\operatorname{Irr}(\tau)$ the set of the irreducible constituents of a character τ. A character $\chi \in \operatorname{Irr}(G)$ is said to be monomial if there exist $H \leq G$ and $\lambda \in \operatorname{Lin}(H)$ such that $\chi = \lambda^G$. A group G is said to be an M-group if all its irreducible characters are monomial. Taketa ([Hu], Satz 5.18.6(b)) has proved that M-groups are solvable. It is natural to suppose that a group G is solvable if the set of its monomial irreducible characters is large. As a corollary of our considerations one obtains that a group G is solvable if all characters of the set $\{\chi \in \operatorname{Irr}(G) \mid \chi(1) < b(G)\}$ are monomial (here $b(G)$ is the maximal degree of irreducible characters of G). Further on if each irreducible character of G is induced from a character of degree at most 2, then G is solvable (Theorem 7).

In the sequel S denotes a nonempty set of simple groups. A group G is said to be an S-group if it is a tower of groups from S. We consider the group $G = \{1\}$ as an S-group. A character $\chi \in \operatorname{Irr}(G)$ is said to be S-monomial if there exist $H \leq G$ and $\lambda \in \operatorname{Irr}(H)$ such that $\chi = \lambda^G$ and $H/\ker\lambda$ is an S-group. The set of all S-monomial characters of G is denoted by $\operatorname{Irr}_S(G)$.

Lemma 1. *Let $N > \{1\}$ be a normal subgroup of a group G. If*

$$|\operatorname{Irr}(G) - \operatorname{Irr}(G/N)| \leq 3,$$

then N is solvable.

Proof. Let $M \subseteq G$. Denote by $k_G(M)$ the number of conjugacy classes of G having nonempty intersections with M. In particular $k(G) = k_G(G)$ is the class number of G. It is known that G is solvable if $k(G) \leq 4$. So if N is nonsolvable, then $N < G$. Since $k(G) = |\operatorname{Irr}(G)|$, by hypothesis

$$k(G/N) + k_G(N) - 1 \leq k(G) \leq k(G/N) + 3$$

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and $k_G(N) \leq 4$. In the sequel we suppose that N is nonsolvable. By Burnside's $\{p, q\}$-theorem one has $k_G(N) \geq 4$. Thus, $k_G(N) = 4$. Let

$$cd(N) = \{\varphi(1) \mid \varphi \in \mathrm{Irr}(N)\} = \{1, d_1, \dots, d_s\}, \qquad 1 < d_1 < \cdots < d_s.$$

Take $\varphi_i \in \mathrm{Irr}(N)$ with $\varphi_i(1) = d_i$, $i = 1, \dots, s$. It follows from Clifford's Theorem that $\langle \varphi_i^G, \varphi_j^G \rangle = 0$ for $i \neq j$, so $s \leq 3$. By the Isaacs Theorem [Is1] we have $s \geq 3$. Thus $s = 3$. Since kernels of all characters from $\mathrm{Irr}(\varphi_i^G)$ do not contain N and $|\mathrm{Irr}(G) - \mathrm{Irr}(G/N)| \leq 3$, it follows that

$$\varphi_i^G = e_i \chi^i, \quad i = 1, 2, 3, \qquad \mathrm{Irr}(G) - \mathrm{Irr}(G/N) = \{\chi^1, \chi^2, \chi^3\}.$$

Let $I_G(\varphi_i)$ be the inertia group of φ_i in G. Set $|G : I_G(\varphi_i)| = t_i$. Then by Clifford's Theorem $\chi^i(1) = e_i t_i d_i$, $|G : N| = e_i^2 t_i$, $i = 1, 2, 3$. Let π be the set of all prime divisors of $|G : N|$. Then $\pi = \pi(e_i t_i)$, $i = 1, 2, 3$ (here $\pi(n)$ is the set of prime divisors of a positive integer n). As $k_G(N) = 4$ it follows from Burnside's $\{p, q\}$-theorem that N is simple. Take $p \in \pi$. If $\chi \in \mathrm{Irr}(G)$ and $p \nmid \chi(1)$, then $N \leq \ker \chi$ by the above. Denote by $G(p')$ the intersection of the kernels of all nonlinear $\chi \in \mathrm{Irr}(G)$ such that $p \nmid \chi(1)$. Obviously $N \leq G(p')$. Since $G(p')$ is p-nilpotent [Be] and N is nonabelian simple, then $p \nmid |N|$. Thus N is a π'-Hall subgroup of G. By the Schur-Zassenhaus Theorem there exists in G a π-Hall subgroup H. Since $N < G$, it follows that $H > \{1\}$. Take x, an element of prime order in H. Since N is not nilpotent, there exists an element $y \in N - \{1\}$ such that $xy = yx$ ([Hu], Hauptsatz 4.8.7(a)). Since any G-conjugate of xy is not contained in $H \cup N$, then

$$k(H) + 3 = k(G/N) + 3 = k(G) \geq k(G/N) + k_G(N) - 1 + 1 = k(H) + 4,$$

which is a contradiction. □

In Remark 1 we use the Tate Theorem (see [Is2, Theorem 6.31]). Let $A^p(G)/G' \in \mathrm{Syl}_{p'}(G/G')$, $P \in \mathrm{Syl}_p(G)$. Obviously, $A^p(P) = P'$. The Tate Theorem asserts that $P \cap A^p(G) = P'$ implies $P \cap O^p(G) = O^p(P) = \{1\}$, where $O^p(G)$ is the unique minimal normal subgroup of G such that $G/O^p(G)$ is a p-group. Assume that N is a normal subgroup of G, and $P \cap N \leq P'$. We shall prove that N has normal p-complement. Without loss of generality we may assume that $G = PN$. Then $P'N = A^p(G)$ and $P \cap P'N = P'$. By the Tate Theorem one obtains $P \cap O^p(G) = O^p(P) = \{1\}$ so $O^p(G)$ is a p'-subgroup. Hence G, and so N, has a normal p-complement.

Remark 1. In the sequel we shall use the following assertion: If N is a nontrivial normal subgroup of G and $|\{\chi(1) | \chi \in \mathrm{Irr}_1(G) - \mathrm{Irr}(G/N)\}| = 1$, then N is solvable. There is an extension of elementary abelian group N of order 2^4 by A_5 which satisfies the above equality. Let us prove this assertion. Assume that N is nonsolvable. Let $\{1\} < N_1 < N$ and N_1 be normal in G. Since $\mathrm{Irr}_1(G/N) \subset \mathrm{Irr}_1(G/N_1)$ (this is due to the fact that the sum of the nonlinear irreducible characters of a nonabelian group is a faithful character), then

$$\mathrm{Irr}_1(G) - \mathrm{Irr}(G/N_1) \subset \mathrm{Irr}_1(G) - \mathrm{Irr}(G/N),$$
$$\mathrm{Irr}_1(G/N_1) - \mathrm{Irr}(G/N) \subset \mathrm{Irr}_1(G) - \mathrm{Irr}(G/N),$$

and it suffices to prove our assertion in the case when N is a minimal normal subgroup of G. Take a nonlinear $\lambda \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(\lambda^G)$. Let p be a prime divisor of $\lambda(1)$. Then $p \mid \chi(1)$ by the Clifford Theorem. By reciprocity N is not contained in $\ker\chi$, i.e. $\chi \in \mathrm{Irr}_1(G) - \mathrm{Irr}(G/N)$. Take $P \in \mathrm{Syl}_p(G)$. Then $P \cap N = P_1 \in \mathrm{Syl}_p(N)$ and P_1 is not contained in P' according to the Tate Theorem (see text before this remark). Therefore there exists a linear character μ of P such that P_1 is not contained in $\ker\mu$. Since $N' = N$, it follows that $N \leq G'$. Since $p \nmid \mu^G(1)$, there exists $\tau \in \mathrm{Irr}(\mu^G)$ such that $p \nmid \tau(1)$. By reciprocity N is not contained in $\ker\tau$, so $\tau \in \mathrm{Irr}_1(G) - \mathrm{Irr}(G/N)$. Therefore $\tau(1) = \chi(1)$, $p \mid \chi(1)$, $p \nmid \tau(1)$, a contradiction. Thus, N is solvable. □

Remark 2. Sometimes in the sequel we will use the following proposition: Let $\chi = \lambda^G \in \mathrm{Irr}(G)$ be faithful, $H \leq G$, $\lambda \in \mathrm{Irr}(H)$, and $H/\ker\lambda$ an S-group. If N is a minimal normal subgroup of G and $N \leq H$, then N is an S-group. This is true since N is not contained in $\ker\lambda$, so N_1 is not contained in $\ker\lambda$ where N_1 is some simple direct factor of N. Then $N_1 \cap \ker\lambda = \{1\}$, so the subnormal subgroup $N_1 \ker\lambda/\ker\lambda$ ($\cong N_1$) of S-group $H/\ker\lambda$ is an S-group as well. Since N is a direct product of groups isomorphic to N_1, it is an S-group, and our claim is proved. □

Remark 2 is due to the referee.

Consider the following property of a group G:

(∗) Whenever $\chi, \tau \in \mathrm{Irr}(G)$ with $\ker\tau = \ker\chi$ and $\chi(1) < \tau(1)$, then χ is S-monomial.

We note that epimorphic images of (∗)-groups are (∗)-groups. Now the number of nonmonomial irreducible characters in (∗)-groups is not bounded.

Theorem 2. *Let S be a set of simple groups containing groups of all prime orders. Then any (∗)-group G is an S-group.*

Proof. Suppose that G is a counterexample of minimal order. If M, N are distinct minimal normal subgroups of G, then MN/M ($\cong N$) as a normal subgroup of an S-group G/M is an S-group (G/M is an S-group by induction, so the claim follows from the Jordan-Holder Theorem). As G/N is an S-group by induction, then G is an S-group, a contradiction. Thus G contains only one minimal normal subgroup N. By assumption N is not an S-group. In particular N is nonsolvable. Since $\bigcap \ker\tau = \{1\}$ (here τ runs over the set $\mathrm{Irr}(G)$), then a group with a unique minimal normal subgroup has a faithful irreducible character. Take in $\mathrm{Irr}(G)$ a faithful character χ of minimal degree.

Suppose that $\chi(1) \geq \tau(1)$ for all faithful $\tau \in \mathrm{Irr}(G)$. Then all faithful irreducible characters of G have the same degree and N is solvable by Remark 1, a contradiction. Thus $\chi(1) < \tau(1)$ for some faithful $\tau \in \mathrm{Irr}(G)$, so χ is S-monomial by hypothesis. Therefore there exist $H \leq G$ and $\lambda \in \mathrm{Irr}(H)$ such that $H/\ker\lambda$ is an S-group and $\chi = \lambda^G$. Since χ is faithful and G is not an S-group, then $H < G$. Take $\xi \in \mathrm{Irr}((1_H)^G)$. Because $(1_H)^G$ is reducible, then $\xi(1) < \chi(1)$. Hence $N \leq \ker\xi$ and $N \leq \ker((1_H)^G) = H_G = \bigcap_{x \in G} H^x \leq H$. In view of Remark 2, N is an S-group, a contradiction. □

Corollary 2.1. *Let S be the set of all groups of prime orders. A group G is solvable if and only if each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) < b(G)$ is S-monomial.*

Corollary 2.2. *Let π be a set of primes. A group G is a π-group if and only if for each $\chi \in \mathrm{Irr}(G)$ there exist $H \leq G$, $\lambda \in \mathrm{Irr}(H)$ such that $H/\ker\lambda$ is a π-group and $\chi = \lambda^G$.*

Let $X(G)$ be the set of all faithful irreducible characters of G, $Y(G) = \{\chi \in X(G) \mid \chi \text{ is } S\text{-monomial}\}$, $V(G) = X(G) - Y(G)$.

Definition. A group G is MS_k if whenever G/N is a monolith, then $|V(G/N)| \leq k$ and $\tau(1) \leq \chi(1)$ for $\tau \in Y(G/N)$, $\chi \in V(G/N)$.

We note that epimorphic images of MS_k-groups are MS_k-groups.

Theorem 3. *Suppose that a set S is such as in Theorem 2. If G is an MS_3-group, then it is an S-group.*

Proof. Assuming that G is a minimal counterexample we see that G contains only one minimal normal subgroup N, G/N is an S-group and N is not an S-group. Therefore $\mathrm{Irr}(G)$ contains a faithful character. Since N is nonsolvable, there exist in $\mathrm{Irr}(G)$ at least four faithful characters by Lemma 1. Take in $\mathrm{Irr}(G)$ a faithful S-monomial character χ of minimal degree (χ exists by condition). By definition $\chi(1) \leq \tau(1)$ for every faithful $\tau \in \mathrm{Irr}(G)$. Then there exist $H \leq G$ and $\lambda \in \mathrm{Irr}(H)$ such that $\chi = \lambda^G$ and $H/\ker\lambda$ is an S-group. Since $(1_H)^G$ is reducible, all its irreducible constituents μ satisfy $\mu(1) < \chi(1)$. Therefore $N \leq \ker(1_H)^G \leq H$, and N is an S-group by Remark 2, a contradiction. □

In particular MS_0-groups are S-groups. A character $\chi \in \mathrm{Irr}(G)$ is said to be monolithic if $G/\ker\chi$ is a monolith. Note that G is an MS_0-group if every monolithic character χ is S-monomial. In particular if every monolithic character χ of G is monomial, then G is solvable. This is a generalization of Taketa's Theorem.

In the same way we may prove the following

Proposition 4. *Let $N > \{1\}$ be a normal subgroup of G. If all characters from $\mathrm{Irr}(G) - \mathrm{Irr}(G/N)$ are monomial, then N is solvable. In particular, the intersection of the kernels of the nonmonomial irreducible characters of G is solvable.*

Proof. Suppose that N is nonsolvable. Let M be the last member of the derived series of N. By assumption $M' = M > \{1\}$. Since $\mathrm{Irr}(G/N) \subseteq \mathrm{Irr}(G/M)$, it follows that $\mathrm{Irr}(G) - \mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$, and it suffices to prove the proposition for M instead of N. In view of Taketa's Theorem one has $M < G$. Since $\bigcap \ker\tau = \{1\}$ where τ runs over the set $\mathrm{Irr}_1(G)$, there is in $\mathrm{Irr}(G) - \mathrm{Irr}(G/M)$ a nonlinear character χ of minimal degree (χ is nonlinear in view of $M = M' \leq G'$). By condition there exist $H \leq G$ and $\lambda \in \mathrm{Lin}(H)$ such that $H/\ker\lambda$ is cyclic and $\chi = \lambda^G$. Take $\psi \in \mathrm{Irr}((1_H)^G)$. Since $\chi(1) > 1$, it follows that $H < G$ and $(1_H)^G$ is reducible. Hence $\psi(1) < \chi(1)$ so that $M \leq \ker\psi$ by the choice of χ. Hence $M \leq \ker((1_H)^G) = H_G \leq H$. Since $H/\ker\lambda$ is solvable and $M' = M$, it follows that $M \leq \ker\lambda$. Therefore $M \leq \ker\chi$—a contradiction with a choice of χ. Let D be the intersection of the kernels of the nonmonomial irreducible characters of G. Then all characters from $\mathrm{Irr}(G) - \mathrm{Irr}(G/D)$ are monomial. Hence, D is solvable.[1] □

[1]Analogously, if all characters from $\mathrm{Irr}(G) - \mathrm{Irr}(G/N)$ are S-monomial, then N is an S-subgroup. In particular, the intersection of the kernels of the non-S-monomial irreducible characters of G is an S-group. Instead of M in the proof, we have to take N^S, the intersection of such normal subgroups A in N such that N/A is an S-group.

Conjecture 1. If all nonmonomial irreducible characters of a group G have the same degree, then G is solvable.

We do not know whether G is solvable if it contains only one nonmonomial irreducible character.

Conjecture 2. Suppose that for every nonlinear $\chi \in \mathrm{Irr}(G)$ there exist $H < G$ (strict inclusion) and $\lambda \in \mathrm{Irr}(H)$ such that $\chi = \lambda^G$. Then G is solvable.

Corollary 5. *Suppose that S is the set of groups of all prime orders. If all $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 3$ are S-monomial, then G is solvable.*

Proof. Take $\chi \in \mathrm{Irr}(G)$. Suppose that $\chi(1) < 4$ and $G/\ker\chi$ is nonsolvable. Then from the classification of linear groups of degrees 2 and 3 it follows that there exists a normal subgroup $A/\ker\chi$ in $G/\ker\chi$ such that G/A is one of the groups $\mathrm{PSL}(2, 5)$, $\mathrm{PSL}(2, 7)$ [B1]. Take $\tau \in \mathrm{Irr}(G/A)$ such that $\tau(1) = 4$ if $G/A = \mathrm{PSL}(2, 5)$ and $\tau(1) = 6$ if $G/A = \mathrm{PSL}(2, 7)$. Since there is not a subgroup H/A in G/A such that $1 < |G : H| \le \tau(1)$, our condition does not hold for G/A and so for G. Thus $G/\ker\chi$ is solvable for all $\chi \in \mathrm{Irr}(G)$ with $\chi(1) < 4$.

Suppose that G is a counterexample of minimal order. Then G contains only one minimal normal subgroup R, G/R is solvable and R is not solvable. Take in $\mathrm{Irr}(G)$ a faithful character χ of minimal degree. By the above $\chi(1) > 3$. Then there exist $H \le G$ and $\lambda \in \mathrm{Irr}(H)$ such that $H/\ker\lambda$ is solvable and $\chi = \lambda^G$. Since for each irreducible constituent τ of $(1_H)^G$ one has $\tau(1) < |G : H| \le \chi(1)$, it follows that $R \le \ker\tau$, so $R \le \ker(1_H)^G \le H$. Since $H/\ker\lambda$ is solvable and $R = R'$, it follows that $R \le \ker\lambda$, so $R \le \ker\chi$, a contradiction. □

It is impossible to replace in Corollary 5 the number 3 by 4. In particular if all $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 3$ are monomial, then G is solvable.

Question. Classify all nonsolvable groups G such that all $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 4$ are monomial.

Denote by $p(G)$ the minimal prime divisor of $|G|$.

In the sequel we use the following known result ([Is2], Problem 3.4):

Lemma 6. *Let G be a nonabelian simple group, p a prime divisor of $|G|$, $P \in \mathrm{Syl}_p(G)$. If $\chi \in \mathrm{Irr}(G)$ is faithful and $\chi(1) = p$, then P is of order p.*

Theorem 7. *Suppose that for each irreducible character χ of G there exist $H \le G$, $\lambda \in \mathrm{Irr}(H)$ such that $\lambda(1) \le p(H)$ and $\lambda^G = \chi$. Then G is solvable.*

Proof. Assume that G is a counterexample of minimal order. Then G contains only one minimal normal subgroup R, G/R is solvable and $R = F_1 \times \cdots \times F_s$ where F_i are isomorphic nonabelian simple groups. Hence G has a faithful irreducible character. Take in $\mathrm{Irr}(G)$ a faithful character χ of minimal degree. By hypothesis there exist $H \le G$, $\lambda \in \mathrm{Irr}(H)$ such that $\lambda(1) \le p = p(H)$ and $\chi = \lambda^G$. To show that $R \le H$, let us consider $(1_H)^G$. If $H = G$, then $R \le H$. So suppose that $H < G$. Then $(1_H)^G$ is reducible. So all irreducible constituents of $(1_H)^G$ are not faithful (their degrees less than $\chi(1)$) and $R \le \ker(1_H)^G \le H$. Since $\chi = \lambda^G$ is faithful, $R = R'$ is not contained in $\ker\lambda$. Hence λ_R has no linear constituents. Therefore $\lambda(1) = p(H)$ and λ_R is irreducible (Clifford). Therefore $p(H) \mid |R|$ and $p(H) = p(R) = p$. Moreover there exists $i \in \{1, \dots, s\}$ such that the restriction of λ on F_i is irreducible. Let P be a Sylow p-subgroup of F_i. Then P is of order p (Lemma 6) and F_i has a normal p-complement by Burnside's normal p-complement theorem. Hence R has a normal p-complement as well, contradicting the equality $R' = R$. □

In particular if every irreducible character of G is induced from a character of degree at most 2, then G is solvable.

Conjecture 3. If any irreducible character of a group G is induced from a character of degree at most 3, then G is solvable.[2]

Conjecture 4. If all irreducible characters of p'-degrees from $\mathrm{Irr}(G)$ are monomial, then G is p-solvable, unless $p < 5$.

Conjecture 5. If all irreducible characters of composite degrees are monomial, then G is solvable.

Conjecture 6. Suppose that every $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is not a power of a fixed prime p is monomial. Then G is solvable.

Let N be a normal subgroup of G. Set $c(N) = |\{\chi(1) \mid \chi \in \mathrm{Irr}_1(G),\ N$ is not contained in $\ker \chi\}|$. If $c(N) = 1$, then N is solvable (Remark 1). Probably if $c(N) = 2$, then N is solvable too. If $N = G = A_5$, then $c(N) = 3$.

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[2]This conjecture is true. Moreover, if any irreducible character of a group G is induced from a character of degree at most 4, then G is solvable, unless $G/S(G) \cong A_5$; here $S(G)$ is the solvable radical of G (see Ya. Berkovich, *On the Taketa Theorem* (to appear)).