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THERE ARE KNOTS WHOSE TUNNEL NUMBERS GO DOWN UNDER CONNECTED SUM

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ABSTRACT. In this paper, we show that there are infinitely many tunnel number two knots K such that the tunnel number of $K\#K'$ is equal to two again for any 2-bridge knot K'.

INTRODUCTION

Let K be a knot in the 3-sphere S^3, and $t(K)$ the tunnel number of K. Here the tunnel number of K is the minimum number of mutually disjoint arcs properly embedded in the exterior of K whose exterior is a handlebody. We call the family of such arcs an unknotting tunnel system for K. In particular, we call it an unknotting tunnel for K, if the family consists of a single arc. On behavior of the tunnel number of knots under connected sum, the most simple case is:

Theorem 1 ([N], [Sc] and [MS]). *Tunnel number one knots are prime.*

And in the previous paper, we have shown:

Theorem 2 ([M1]). *Let K_1 and K_2 be non-trivial knots in S^3, and suppose $t(K_1\#K_2) = 2$. Then:*

(1) *if neither K_1 nor K_2 is a 2-bridge knot, then $t(K_1) = t(K_2) = 1$ or,*

(2) *if one of K_1 and K_2, say K_1, is a 2-bridge knot, then $t(K_2) \leq 2$ and K_2 is prime.*

In this paper, we show :

Theorem 3. *There are infinitely many tunnel number two knots K such that $t(K\#K') = 2$ for any 2-bridge knot K'.*

1. KNOT K_n

Let n be an integer (> 1) and K_n the knot illustrated in Figure 1.

To prove Theorem 3, we show that (1) : $t(K_n) = 2$, (2) : $t(K_n\#K) = 2$ for any 2-bridge knot K and (3) : K_n and $K_{n'}$ are mutually different types if n and n' are mutually different integers (> 1).

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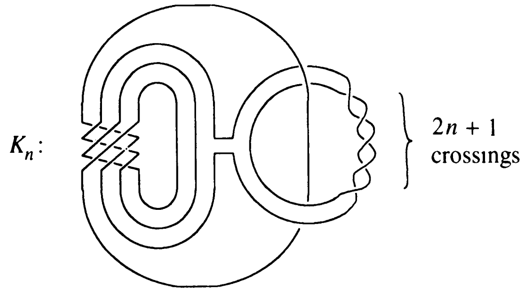

FIGURE 1

Lemma 1. *$t(K_n) \leq 2$ and $t(K_n \# K) \leq 2$ for any 2-bridge knot K.*

Proof. Let γ_1 and γ_2 be the two arcs indicated in Figure 2-(1). Then by the deformation illustrated in Figure 2-(1) through Figure 2-(6), and since the arc ρ indicated in Figure 2-(6) is an unknotting tunnel for the (4, 3)-torus knot (cf. [BRZ]), we see that $cl(E(K_n \# K) - N(\gamma_1 \cup \gamma_2))$ is a genus three handlebody, where $E(K_n \# K) = cl(S^3 - N(K_n \# K))$ and $N(\cdot)$ denotes a regular neighborhood. This shows that $\{\ \gamma_1,\ \gamma_2\ \}$ is an unknotting tunnel system for $K_n \# K$. Since any 2-bridge knot has such a projection that each block has even crossings, we have $t(K_n \# K) \leq 2$ for any 2-bridge knot K. Moreover, since a trivial knot has a 2-bridge decomposition, this inequality implies $t(K_n) \leq 2$. This completes the proof of the lemma.

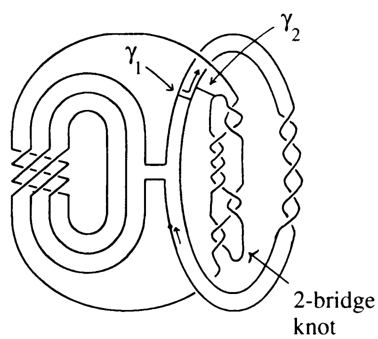

FIGURE 2-(1)

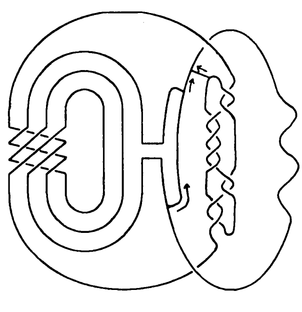

FIGURE 2-(2)

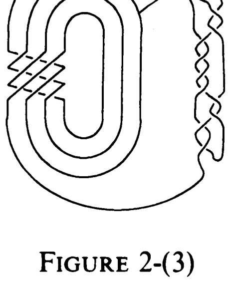

FIGURE 2-(3)

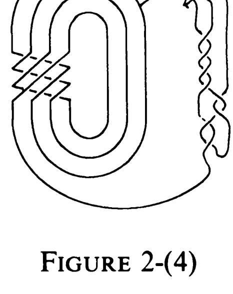

FIGURE 2-(4)

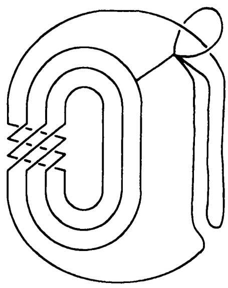

FIGURE 2-(5)

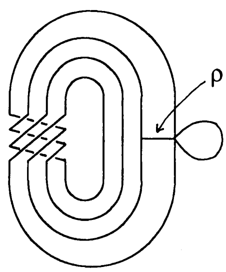

FIGURE 2-(6)

2. TANGLE T_m

Let m be an integer (> 0) and $(B, s \cup t)$, say T_m, the 2-string tangle illustrated in Figure 3, where B is a 3-ball, s is the trivial arc properly embedded in B and t is the knotted arc properly embedded in B. Since there is a disk properly embedded in B which intersects $s \cup t$ in two points and splits T_m into two 2-string trivial tangles, we have the next lemma.

Lemma 2. *The 2-fold branched covering space of B along $s \cup t$, say $\Sigma_2(T_m)$, is a Seifert fibered space over a disk with two exceptional fibers, and the Seifert invariants are $\frac{1}{2}$ and $\frac{-1}{2m+1}$. Hence if $m \neq m'$, then T_m is not homeomorphic to $T_{m'}$* (cf. [Se]).

Let S be the 2-sphere indicated in Figure 4 (on the next page) and B_1, B_2 the 3-balls bounded by S. Put $B_i \cap K_n = s_i \cup t_i$ $(i = 1, 2)$, where s_i is a trivial arc and t_i is a knotted arc. And let α_i be the arc in ∂B_i indicated in Figure 5-(1) and in Figure 6-(1) (on the next page) connecting two points in $\partial(s_i \cup t_i)$. Then by the deformation illustrated in Figure 5-(1) through 5-(4) and in Figure 6-(1) through 6-(2), we have the next lemma.

Lemma 3. *$(B_1, s_1 \cup t_1)$ is homeomorphic to T_1, $(B_2, s_2 \cup t_2)$ is homeomorphic to T_n and α_1 is identified with α_2.*

Lemma 4. *The 2-fold branched covering space of S^3 along K_n, say $\Sigma_2(K_n)$, is a union of two Seifert fibered spaces $\Sigma_2(T_1)$ and $\Sigma_2(T_n)$, and is not a Seifert fibered space. Hence the preimage of S is a torus which gives the torus decomposition of $\Sigma_2(K_n)$* (cf. [JS] and [Jo]).

Proof. Put $T = \partial\Sigma_2(T_1) = \partial\Sigma_2(T_2)$. Then T is a separating incompressible torus in $\Sigma_2(K_n)$.

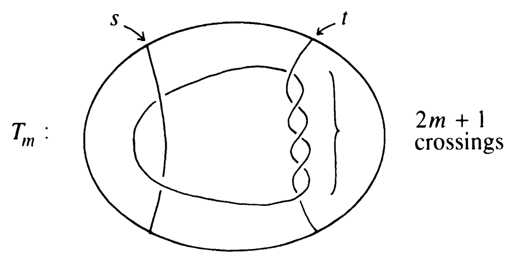

FIGURE 3

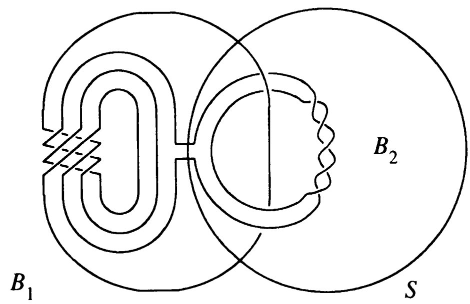

FIGURE 4

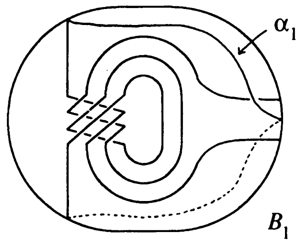

FIGURE 5-(1)

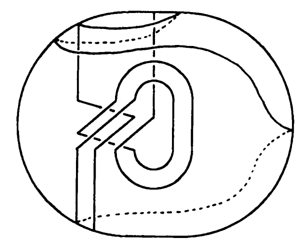

FIGURE 5-(2)

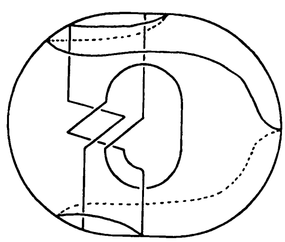

FIGURE 5-(3)

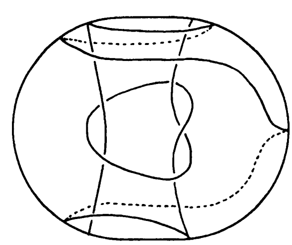

FIGURE 5-(4)

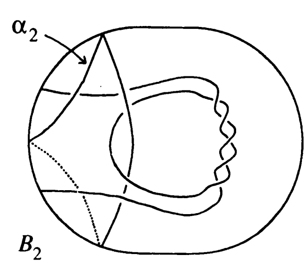

FIGURE 6-(1)

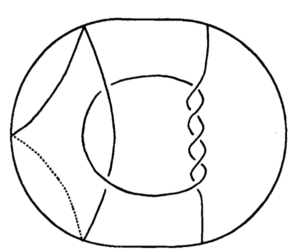

FIGURE 6-(2)

Suppose $\Sigma_2(K_n)$ is a Seifert fibered space. Then by Theorem VI.34 of [Ja], T is saturated in some Seifert fibration of $\Sigma_2(K_n)$, or splits $\Sigma_2(K_n)$ into two twisted I-bundles over a Klein bottle. By Theorem VI.17 of [Ja] and Lemma 2, both $\Sigma_2(T_1)$ and $\Sigma_2(T_n)$ admit unique Seifert fibrations, and $\Sigma_2(T_1)$ is not a twisted I-bundle over a Klein bottle. Hence the Seifert fibration of $\Sigma_2(T_1)$ has to extend to the Seifert fibration of $\Sigma_2(T_n)$. However, it does not extend because, by the proof of Lemma 3, the preimage of α_2 is a regular fiber of $\Sigma_2(T_n)$, the preimage of α_1 is not a regular fiber of $\Sigma_2(T_1)$ and α_1 is identified with α_2. This is a contradiction and completes the proof of the lemma.

The next lemma is an immediate consequence of Birman-Hilden and Viro's result, that is, any genus two orientable closed 3-manifold has an orientation preserving involution which preseves each genus two handlebody setwise (cf. [BH] and [V]).

Lemma 5. *If a knot K has tunnel number one, then there is an orientation preserving involution h of S^3 such that $h(K) = K$ and the fixed point set of h intersects K in two points.*

We note here that the above involution h reverses the orientation of K.

3. Proof of Theorem 3

Suppose $t(K_n) \leq 1$. Then there is an involution h as in Lemma 5. Let T be the preimage of S in $\Sigma_2(K_n)$ and $\tilde{h}$ the lift of h, i.e. $\tilde{h}$ is a self-homeomorphism of $\Sigma_2(K_n)$ with $h \circ p = p \circ \tilde{h}$, where $p : \Sigma_2(K_n) \to S^3$ is the projection. Then by the uniqueness of torus decomposition, $\tilde{h}$ is isotopic to a self-homeomorphism $\tilde{h}_*$ of $\Sigma_2(K_n)$ such that $\tilde{h}_*(T) = T$. Moreover by [BS], we can choose the isotopy to be equivariant with p. Hence we may assume that $h(S) = S$.

Since $n > 1$, by Lemma 2 we have $h(B_1) = B_1$ and $h(B_2) = B_2$. Moreover since s_i is a trivial arc and t_i is a knotted arc $(i = 1, 2)$, we have $h(s_1) = s_1$, $h(t_1) = t_1$, $h(s_2) = s_2$ and $h(t_2) = t_2$. However, since h reverses the orientation of the arcs, each arc has one fixed point. This shows that the fixed point set of h intersects K_n in four points. This contradicts Lemma 5 and completes the proof of $t(K_n) = 2$. And by Theorem 1, we have $t(K_n \# K) = 2$.

Let n' be an integer (> 1) different from n, and let $(B'_1, s'_1 \cup t'_1) \cup (B'_2, s'_2 \cup t'_2)$ be the tangle decomposition of $(S^3, K_{n'})$ corresponding to that of (S^3, K_n). Suppose there is a homeomorphism f of (S^3, K_n) to $(S^3, K_{n'})$. Then by the same argument as above, we may assume that $f(B_1) = B'_1$ and $f(B_2) = B'_2$. Hence T_n is homeomorphic to $T_{n'}$. This contradicts Lemma 2 and completes the proof of Theorem 3.

Remark. (1) Since the tangle T_m is prime, the fact " $t(K_n) = 2$ " is an immediate consequence of Theorem 2.3 of [Sc], that is " Tunnel number one knots are doubly prime ". In this paper, we showed that K_n is not strongly invertible.

(2) Let K be a tunnel number two knot. In the forthcoming paper [M2], we show that if $t(K \# K') = 2$ for some 2-bridge knot K', then $t(K \# K') = 2$ for any 2-bridge knot K'.

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